Semi-modular forms

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$GL_2(\mathbb{Z})$ and some “special" subgroups

Special 2×2 matrices encode “nice” transformations in $\mathbb{C}$:

Let $GL_2(\mathbb{Z})$ denote the set of 2×2 invertible matrices with integer entries (“general linear group”).

Let $SL_2(\mathbb{Z})$ denote the subset of $GL_2(\mathbb{Z})$ with determinant 1.

Let $PSL_2(\mathbb{Z}) \subset GL_2(\mathbb{Z})$ with determinant ± 1 (“modular group”).

Functions of $z \in \mathbb{H}$ invariant under (weighted) actions of $SL_2(\mathbb{Z})$ are “modular forms”.

$GL_2(\mathbb{Z})$ and some "special" subgroups

The general linear group has canonical generators:

$$GL_2(\mathbb{Z}) = \langle T, U, V \rangle$$

where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The special linear group has canonical generators:

$$SL_2(\mathbb{Z}) = \langle S, T \rangle$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Modular forms

A modular form of weight k is an analytic function $f : \mathbb{H} \to \mathbb{C}$ such that

1. $f(S \cdot z) = z^k f(z)$ (weighted invariance under S)
2. $f(T \cdot z) = f(z)$ (invariance under T)

- Modular forms central to contemporary number theory
- Connect algebra, compl. analysis, combin., physics, ...
- Hardy-Ramanujan circle method, proof of FLT
- Interesting variants: quasi-mod. forms, mock mf's, qmf's, ...

Modular forms

An (entire) modular form of weight k is an analytic function $f : \mathbb{H} \to \mathbb{C}$ such that

1. $f(S \cdot z) = z^k f(z)$ (weighted invariance under S)
2. $f(T \cdot z) = f(z)$ (invariance under T)
3. f remains bounded as $z \to i\infty$.

Example: The Eisenstein series of weight $2k, k > 1, z \in \mathbb{H}$:

$$G_{2k}(z) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(mz+n)^{2k}}.$$

Semi-modular forms

Question: Given $SL_2(\mathbb{Z}) \subset GL_2(\mathbb{Z})$, it is also possible to write

$$GL_2(\mathbb{Z}) = \langle S, T, M \rangle$$

where M is some other matrix.

What's the deal with M? What about $\langle M, S \rangle, \langle M, T \rangle$?

Semi-modular forms

A **semi-modular form** is a “half modular” analytic function $f : \mathbb{H} \to \mathbb{C}$ where **two** of the following three relations hold:

$$f(M \cdot z) = f(z)$$

$$f(T \cdot z) = f(z)$$

$$f(S \cdot z) = z^k f(z)$$

for some k, where M is such that $GL_2(\mathbb{Z}) = \langle S, T, M \rangle$.

Semi-modular forms

Example: Take $M_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (even function matrix):

$$f(M_0 \cdot z) = f(z) \text{ takes the form } f(-z) = f(z).$$

Periodic even fctns with period 1 are invariant on $\langle M_0, T \rangle$, so

$$f(z) = \cos(2\pi z)$$

is a semi-modular form. **Are there fctns invariant on $\langle M_0, S \rangle$?**

Semi-modular forms

Question: Where does weighted invar. under S come from in

$$G_k(\tau) = \sum_{\substack{a,b \in \mathbb{Z} \\ (a,b) \neq (0,0)}} (a\tau + b)^{-k}?$$

Of course:

$$\begin{aligned} G_k(-1/\tau) &= \sum_{\substack{a,b \in \mathbb{Z} \\ (a,b) \neq (0,0)}} (-a/\tau + b)^{-k} \\ &= \tau^k \sum_{\substack{a,b \in \mathbb{Z} \\ (a,b) \neq (0,0)}} (b\tau - a)^{-k} = \tau^k G_k(\tau). \end{aligned}$$

We copy this for $\langle M_0, S \rangle$ w/ symmetries of set $\mathcal{P}$ of partitions.

Partition Eisenstein series

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition (multiset of positive integers) of size $|\lambda| = \sum \lambda_i = n$, and $\mathcal{P}_n$ be the set of all partitions of size n.

Let $\pi : \mathcal{P}_n \to \mathcal{P}_n$ be a bijection (conjugation, odd/distinct, etc).

Then for any function $f : \mathcal{P}_n \to \mathbb{C}$:

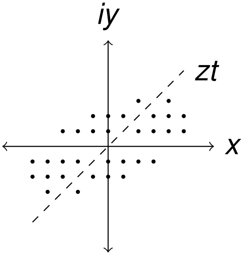
$$\sum_{\lambda \in \mathcal{P}_n} f(\lambda) = \sum_{\lambda \in \mathcal{P}_n} f(\pi(\lambda)).$$

Embedding partitions into $\mathbb{C}$

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ and $z \in \mathbb{C} \setminus \mathbb{R}$, define

$$\mathscr{F}(\lambda, z) = \{a + bz : 1 \leq |a| \leq r \text{ and } 1 \leq |b| \leq \lambda_{|a|}\}.$$

For $\lambda = (3, 2, 2, 1)$ and $z = 1 + i$:



Summing over the points in the lattice

For a partition λ, $z \in \mathbb{C} \setminus \mathbb{R}$, and $k \geq 0$, define

$$f_k(\lambda, z) \;:=\; \frac{1}{4} \sum_{\omega \in \mathscr{F}(\lambda, z)} \omega^{-k}.$$

1. If $k = 0$ then $f_k(\lambda, z) = |\lambda|$ (thus f_k represents a generalization of partition size).
2. If k is odd then $f_k(\lambda, z) = 0$.
3. If k is even then $f_k(\lambda, z)$ is an even function of z.
4. Moreover, if $\overline{\lambda}$ is the conjugate of λ, we have

$$z^{2k} f_{2k}(\lambda, z) = f_{2k}\left(\overline{\lambda}, -\frac{1}{z}\right).$$

$z \mapsto -1/z$ induces partition conjugation!

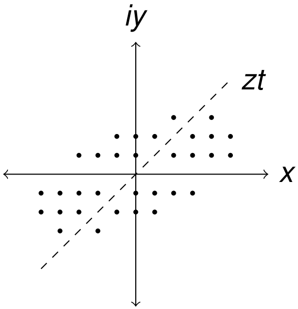
Again let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, and $k > 0$. Then

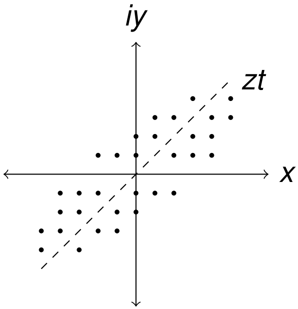
$$f_{2k}(\lambda, z) = \frac{1}{2} \sum_{1 \le a \le r} \sum_{1 \le b \le \lambda_a} \left[(a + bz)^{-2k} + (a - bz)^{-2k}\right]$$

$$\begin{aligned}
f_{2k}(\lambda, -1/z) &= \frac{1}{2} \sum_{1 \le a \le r} \sum_{1 \le b \le \lambda_a} \left[(a - b/z)^{-2k} + (a + b/z)^{-2k}\right] \\
&= \frac{z^{2k}}{2} \sum_{1 \le a \le r} \sum_{1 \le b \le \lambda_a} \left[(az - b)^{-2k} + (az + b)^{-2k}\right] \\
&= \frac{z^{2k}}{2} \sum_{1 \le b \le \lambda_1} \sum_{1 \le a \le \overline{\lambda}_b} \left[(b + az)^{-2k} + (b - az)^{-2k}\right] \\
&= z^{2k} f_{2k}(\overline{\lambda}, z)
\end{aligned}$$

Conjugate partitions have dual lattices

Lattice for $\lambda = (3, 2, 2, 1)$ (left) and $\overline{\lambda} = (4, 3, 1)$ (right) with $z = 1 + i$.





Summing over partitions of n

The notation $\lambda \vdash n$ means $|\lambda| = n$. Define

$$g_k(n, z) := \sum_{\lambda \vdash n} f_k(\lambda, z).$$

Theorem (Just-S.)

$g_k(n, z)$ is a semi-modular form with respect to $\langle M_0, S \rangle$.

Proof. Clearly g_{2k} is an even function of z. Moreover,

$$\begin{aligned} g_{2k}(n, -1/z) &= \sum_{\lambda \vdash n} f_{2k}(\lambda, -1/z) \\ &= z^{2k} \sum_{\lambda \vdash n} f_{2k}(\overline{\lambda}, z) \\ &= z^{2k} \sum_{\lambda \vdash n} f_{2k}(\lambda, z) \;=\; z^{2k} g_{2k}(n, z). \end{aligned}$$

q-series generating function of partition Eisenstein series

For $|q| < 1$ define

$$\mathscr{G}_k(z) \;=\; \mathscr{G}_k(z,q) \;:= \sum_{n\geq 1} g_k(n,z)q^n = \sum_{\lambda\in\mathcal{P}} f_k(\lambda,z)q^{|\lambda|}.$$

Theorem (Just-S.)

$\mathscr{G}_k(z)$ is semi-modular with respect to $\langle M_0, S\rangle$.

Proof. $\mathscr{G}_k(z)$ inherits semi-modularity from $g_k(n,z)$. Since $|1+z| \leq w$ for every $w \in \mathscr{F}(\lambda,z)$, we have

$$|g_k(n,z)| \leq np(n)|1+z|^{-k}.$$

Convergence for $|q| < 1$ follows from conver. of $\sum_{n\geq 1} np(n)q^n$.

Not a total shocker that partitions connect to semi-modularity:

- partition gen. function is essentially modular
- modularity used to prove $p(n)$ asymptotic
- q-bracket of Bloch-Okounkov produces quasi-mfs

Question: Semi-modularity from other areas in mathematics?

Fibonacci-Eisenstein series

Let F_n be the nth Fibonacci number, $F_0 := 0$, $F_{-n} = (-1)^{n-1}F_n$. We define

$$\mathscr{F}_k(z) = \sum_{n\in\mathbb{Z}} (F_n z + F_{n-1})^{-k},$$

a Fibonacci variant of the Eisenstein series.

Analytic properties of Fibonacci-Eisenstein series

- $\mathscr{F}_k(z)$ is absolutely convergent and holomorphic for all $z \in \mathbb{H}$ and $k \geq 4$,
- $\mathscr{F}_k(z)$ has simple poles on $z = F_n/F_{n-1}$ for all $n \in \mathbb{Z}$ of order k and essential singularities at ϕ and $-1/\phi$ where ϕ is the golden ratio, $\phi = \lim\limits_{n\to\infty} \frac{F_n}{F_{n-1}}$.

Fibonacci-Eisenstein series

Let $M_1 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ (mirror symmetry around $z = 1/2$).
Since $U = M_1 TS$, $V = SM_1 TS^3$ we have $GL_2(\mathbb{Z}) = \langle M_1, S, T \rangle$.

Theorem (Akande-S.)

For all $z \in \mathbb{C}$ where $z \neq \phi, -1/\phi, F_n/F_{n-1}$ for all $n \in \mathbb{Z}$ and all $k \in \mathbb{Z}$:

- $\mathscr{F}_{2k}(S \cdot z) = z^{2k} \mathscr{F}_{2k}(z)$
- $\mathscr{F}_{2k}(M_1 \cdot z) = \mathscr{F}_{2k}(z)$.

Thus $\mathscr{F}_{2k}(z)$ is semi-modular with respect to $\langle M_1, S \rangle$.

Proof. F_n recursion gives $\mathscr{F}_{2k}(1+z) = \mathscr{F}_{2k}(-z)$. Also,

$$\begin{aligned}
\mathscr{F}_{2k}(-z) &= \sum_{n\in\mathbb{Z}} (F_n(-z) + F_{n-1})^{-2k} \\
&= z^{-2k} \sum_{n\in\mathbb{Z}} (-F_n + F_{n-1}(1/z))^{-2k} \\
&= z^{-2k} \sum_{n\in\mathbb{Z}} ((-1)^n F_{-n} + (-1)^{n-2} F_{-(n-1)}(1/z))^{-2k} \\
&= z^{-2k} \sum_{-n\in\mathbb{Z}} ((-1)^n F_{-n} + (-1)^{n-2} F_{-(n-1)}(1/z))^{-2k} \\
&= z^{-2k} \sum_{n\in\mathbb{Z}} (F_n + F_{n+1}(1/z))^{-2k} = z^{-2k} \mathscr{F}_{2k}(1/z).
\end{aligned}$$

Taking $z \mapsto -z$ proves the theorem.

Lucas sequences

A Lucas sequence is defined by the recursion

$$L_n(a,b) = aL_{n-1}(a,b) - bL_{n-2}(a,b).$$

When $L_0(a,b) = 0, L_1(a,b) = 1 \to$ Lucas seq. of the first kind.
When $L_0(a,b) = 2, L_1(a,b) = a \to$ Lucas seq. of the 2nd kind.

$L_n(1,-1)$ of the first and second kind are the Fibonacci and Lucas numbers respectively.

Lucas sequence Eisenstein series

$$\mathscr{L}_{a,b,m}(z) = \sum_{j=-\infty}^{\infty} \left(L_j(a,b)z + L_{j-1}(a,b)\right)^{-m}$$

Lucas sequence Eisenstein series

For $a \neq 0$, let $M_a = \begin{pmatrix} -1 & a \\ 0 & 1 \end{pmatrix}$ (symmetry around $z = a/2$).

Since $M_a T^a = M_1 T$ if $a > 0$ and $T^{-a} M_a = M_1 T$ if $a < 0$, then $GL_2(\mathbb{Z}) = \langle M_a, S, T \rangle$.

Theorem (Akande-S.)

We have

$$\mathscr{L}_{a,-1,2k}(S \cdot z) = z^{2k} \mathscr{L}_{a,-1,2k}(z),$$

$$\mathscr{L}_{a,-1,2k}(M_a \cdot z) = \mathscr{L}_{a,-1,2k}(z).$$

Then $\mathscr{L}_{a,-1,2k}(z)$ is semi-modular w/ respect to $\langle M_a, S \rangle$.

Outstanding questions

1. What is the structure of the space of semi-modular forms?
2. Connection to modular forms/quasimodular forms?
3. Other applications of Ferrers-Young lattices as combinatorial objects?
4. Do there exist other complementary matrices **not** of form $M_a, a \in \mathbb{Z}$, that admit nice semi-modular forms?

Thank you for listening!