Partition-theoretic model of prime distribution

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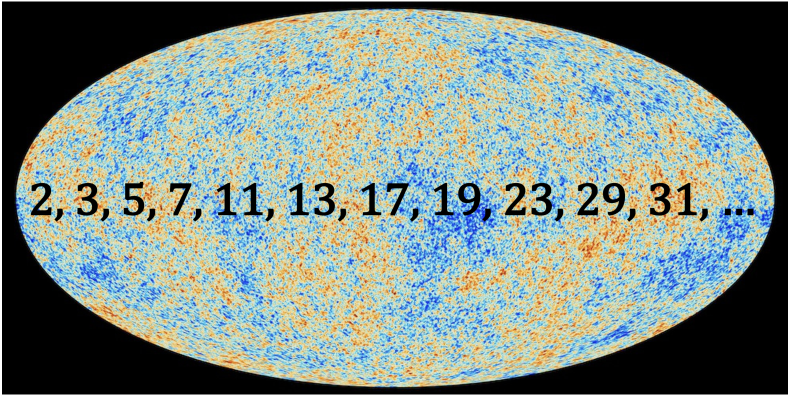
Joint work with Aidan Botkin, Madeline L. Dawsey, David J. Hemmer and Matthew R. Just, featuring computations by Eli DeWitt

Partitions and q-Series Seminar, January 22, 2026

Primordial mystery



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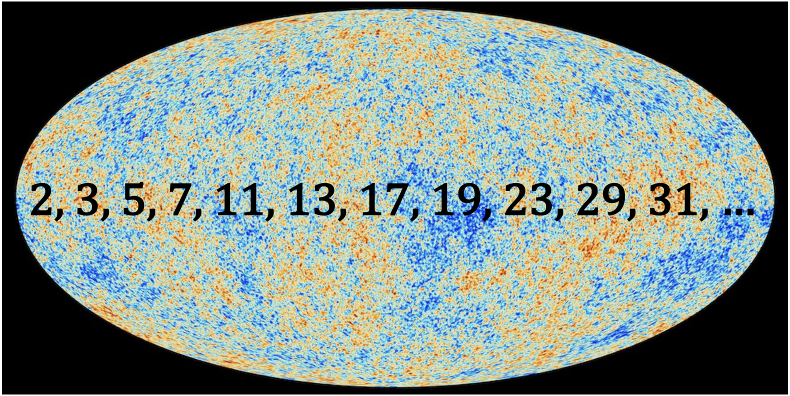

Prime numbers have been around forever... yet aspects of their behavior are beyond human understanding.

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Terence Tao: “[Random] models are so effective... that analytic number theory is in the curious position of being able to *confidently predict the answer to a large proportion of the open problems in the subject, whilst not possessing a clear way forward to rigorously confirm these answers*!”

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10	4	4.34...	6.16...	4	4	4
100	25	21.71...	30.12...	27	26	27
1000	168	144.76...	177.60...	184	168	171
10,000	1229	1085.73...	1246.13...	1352	1212	1233
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1,000,000	78,498	72,382.41...	78,627.54...	86,739	77,322	78,740

TABLE 1. Comparing estimates for $\pi(n)$

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We derive a *deterministic* model of primes from facts about *partitions*. Let $\pi(n)$ denote the $\#$ of primes $\leq n$.

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Note: Model 2* is *almost exact* at small numbers.

Theory of addition

Multiplicative theory of additive partitions

Theory of addition

- theory of *partitions*

Theory of addition

- theory of *partitions*
- beautiful generating functions
- surprising bijections
- Ramanujan congruences
- connections across mathematics, phys sciences, CS

Theory of multiplication

Theory of multiplication

- primes
- divisors
- Euler phi function $\varphi(n)$, Möbius function $\mu(n)$
- arithmetic functions, Dirichlet convolution
- zeta functions, Dirichlet series, L-functions

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- Objects in multiplic. number theory

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- Expect partition properties $\rightarrow$ influence on integers

Partition notations

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- See “The product of parts or ‘norm’” (Schneider-Sills)

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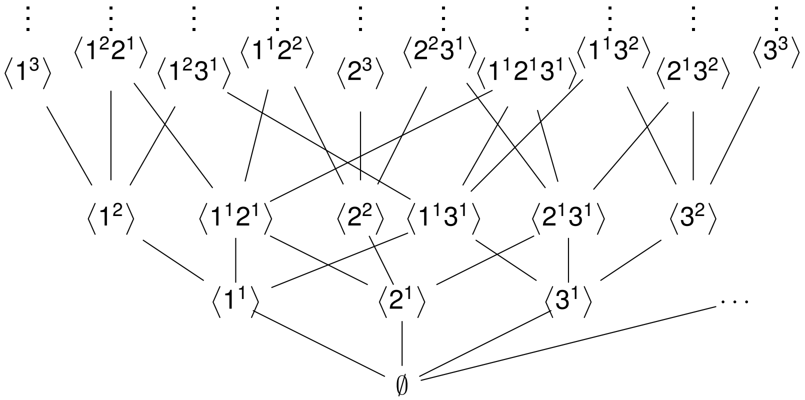
Theorem (Dawsey-Just-S., 2022)

Map $\widehat{N} : \mathcal{P} \to \mathbb{N}$ is an isom. of monoids under multiplic.

Isomorphism of monoids $\mathcal{P} \cong \mathbb{N}$ (Dawsey-Just-S., 2022)



Lattice of partitions ordered by multiset inclusion

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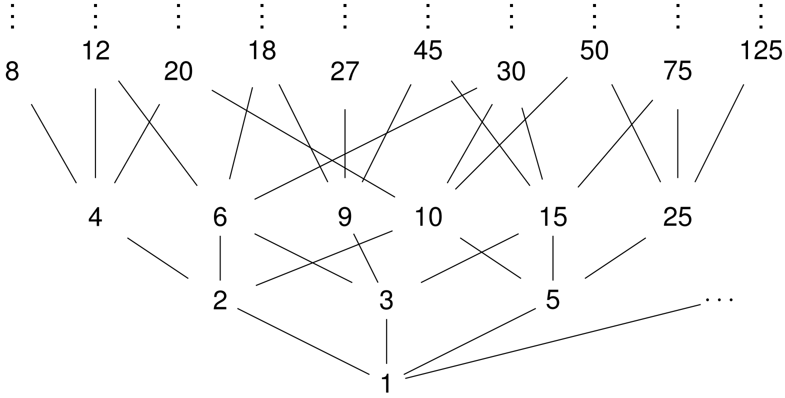

Lattice of integers ordered by divisibility

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Approach: Exploit discrepancy between $N(\lambda)$ and $\widehat{N}(\lambda)$.

Discrepancy between the norm and supernorm

Partition λ with no part equal to 1 respects

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Moreover, when $N(\lambda) \geq 5$, if λ has parts $\neq 1$ then

$$N(\lambda) < p_{N(\lambda)} \le \widehat{N}(\lambda) \le N(\lambda)^{\log 3/\log 2}$$

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so the supernorm lies in a “small” interval $\ge p_{N(\lambda)}$, and

$$\widehat{N}(\lambda) \ >\ N(\lambda) \cdot \prod_{i\ge 2} (\log i)^{m_i(\lambda)}$$

so $\widehat{N}(\lambda)$ can be closer to $N(\lambda)$ if there are fewer parts.

Discrepancy between the norm and supernorm

These inequalities suggest the following:

1. Partitions λ with a smaller number $\ell(\lambda)$ of parts $\neq 1$ should have supernorms closer to their norms.
2. Partitions λ with same norm $N(\lambda) = n$ and parts $\neq 1$ should have supernorms of comparable magnitudes.
3. Partitions λ with same norm $N(\lambda) = n \geq 5$ and parts $\neq 1$ respect the inequality $p_n \leq \widehat{N}(\lambda)$ (bound on p_n).

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Combining these bullet points, we make kind of an extreme simplification.

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Assumption

Assume all odd integers in the interval $[p_n, p_{n+1}),\ n \geq 2$, are images of partitions with norm n, with parts $\neq 1$, and with only one or two parts under $\widehat{N}$.

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Since there are $d(n)$ partitions of norm n into 1 or 2 parts,

$$p_{n+1} - p_n \ = \ 2\left\lceil \frac{d(n)}{2} \right\rceil,$$

where $\lceil x \rceil$ is ceiling function (factor of 2 includes even $\#$'s)

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Model 1

The prime numbers are modeled by the sequence $p_1, p_2, p_3, \ldots$, of positive integers defined by $p_1 = 2$, and for $n \geq 2$ by the relation

$$p_n = 1 + 2\sum_{k=1}^{n-1} \left\lceil \frac{d(k)}{2} \right\rceil .$$

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This gives $2, 3, 5, 7, 11, 13, 17, 19, 23, \mathbf{27}, \ldots,$ then wrong. But it **almost** gives the correct sequence of **prime gaps.**

Partition model of primes (Botkin-Dawsey-Hemmer-Just-S.)

n	p_n	$p_{n+1} - p_n$ (Actual)	$p_{n+1} - p_n$ (Model 1)
1	2	1	1
2	3	2	2
3	5	2	2
4	7	4	4
5	11	2	2
6	13	4	4
7	17	2	2
8	19	4	4
9	23	6	4
10	29	2	4
11	31	6	2
12	37	4	6
13	41	2	2
14	43	4	4
15	47	6	4
16	53	6	6
17	59	2	2
18	61	6	6
19	67	4	2
20	71	2	6
21	73	6	4
22	79	4	4
23	83	6	2
24	89	8	8
25	97	4	4

TABLE 2. Comparing actual prime gaps to predictions from Model 1; we highlight entries where the prediction for the nth prime gap is off

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16 out of 25 gaps are correct, including most twin primes.

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Model 1 gives main term of the prime number theorem:

$$p_n \sim n \log n \quad \text{as} \quad n \to \infty.$$

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Proof: From Model 1,

$$p_n \; = \; 1 + 2 \sum_{k=1}^{n-1} \left\lceil \frac{d(k)}{2} \right\rceil \; \sim \sum_{1 \leq i \leq n-1} d(i) \sim \; n \log n$$

by estimate of Dirichlet.

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Proof: Model 1 says $p_{n+1} - p_n = 2\left\lceil \frac{d(n)}{2} \right\rceil = 2$ if n is prime.

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But we cannot take this too seriously since the model represents an underestimate of prime gaps.

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- We do not have conclusive evidence this is due to prime indices (not some other reason).

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Define *Model 1 merit* of the nth *modeled* prime gap by $M_1(n) = \frac{d(n)}{\log n}$, noting $\log n$ is average size of $d(n)$.

Measure deviation of real/modeled gaps from average.

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Question: *Do the statistics increase/decrease together?*

Results of computations on MTU HPC Shared Facility

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Up to one million, a slight majority of prime gaps *do* increase or decrease with the divisor function.

We do not know another reason that prime gaps and $d(n)$ should increase/decrease with the same frequency.

Partition model of primes (Botkin-Dawsey-Hemmer-Just-S.)

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Also subtract a factor of $2\gamma n$ for computational reasons (simplifies integral expressions of the estimate for p_n).

Model 2. *The prime numbers are the sequence $p_1, p_2, p_3, \ldots,$ of positive integers defined by $p_1 = 2$, and for $n \geq 2$ by the relation*

$$p_n \;=\; 1 \;+\; 2\sum_{k=1}^{n-1}\left\lceil \frac{d(k)}{2} \right\rceil \;+\; \lfloor \pi_2(p_{n-1}) - 2\gamma n \rfloor \,,$$

where $\gamma = 0.5772\ldots$ is the Euler-Mascheroni constant.

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Model 2* (computational version). *The primes are modeled by the sequence $p_1, p_2, p_3, \ldots,$ defined by $p_1 = 2$, and for $n \geq 2$,*

$$p_n \;=\; 1 \;+\; 2\sum_{k=1}^{n-1}\left\lceil \frac{d(k)}{2} \right\rceil \;+\; \lfloor n \log\log n - 2\gamma n \rfloor\,.$$

Partition model of primes (Botkin-Dawsey-Hemmer-Just-S.)

Compare and contrast Model 2 and 2* estimates for $\pi(n)$:

n	$\pi(n)$	$n/\log n$	$\mathrm{li}(n)$	Model 1	Model 2	Model 2*
10	4	4.34...	6.16...	4	4	4
100	25	21.71...	30.12...	27	26	27
1000	168	144.76...	177.60...	184	168	171
10,000	1229	1085.73...	1246.13...	1352	1212	1233
100,000	9592	8685.88...	9629.80...	10,602	9435	9618
1,000,000	78,498	72,382.41...	78,627.54...	86,739	77,322	78,740

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TABLE 1. Comparing estimates for $\pi(n)$

At small values: almost exact. At large values: gives PNT.

Partition model of primes (continued)

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n	$\pi(n)$	$n/\log n$	Model 1	Model 2	Model 2*	Model 3 $r=6,\ t=0.11$
10	4	4.34...	4	4	4	4
100	25	21.71...	27	26	27	25
1000	168	144.76...	184	168	171	168
10,000	1229	1085.73...	1352	1212	1233	1228
100,000	9592	8685.88...	10,602	9435	9618	9592
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TABLE 1. Comparing estimates for $\pi(n)$; Model 3 depends on parameters r, t

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Thank you for listening!