

# **Under construction: A multiplicative theory of integer partitions**

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# Additive number theory

### Patterns and interconnections

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- theory of *partitions*

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- theory of *partitions*
- beautiful generating functions
- surprising bijections
- Ramanujan congruences
- combinatorics, algebra, analytic num. theory, mod. forms, stat. phys., QT, string theory, chemistry, ...

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- connected analysis to partitions

# Multiplicative number theory

## Multiplicative number theory

**I.e., most of classical number theory**

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- primes
- divisors
- Euler phi function  $\varphi(n)$ , Möbius function  $\mu(n)$
- arithmetic functions, Dirichlet convolution
- zeta functions, Dirichlet series, L-functions

### Euler



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- explicit zeta values  $\rightarrow$  compute even powers of  $\pi$ :

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \dots$$

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- Proofs feel similar (multiply geometric series)

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Are other thms. in arithmetic *images in prime partitions* of combinatorial/set-theoretic meta-structures?

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Is there an *algebra of partitions* generalizing arithmetic in integers (i.e., prime partitions)? **I want to believe.**

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- Define  $\ell(\emptyset) = |\emptyset| = m_i(\emptyset) = 0, \quad \emptyset \vdash 0$ .

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- See “The product of parts or ‘norm’ etc.” (S-Sills)

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- Identity is  $\emptyset$
- Partitions into one part are like primes, FTA holds

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- Replacing  $\mathcal{P}$  with  $\mathcal{P}_{\mathbb{P}}$  reduces to  $\mu(N(\lambda))$ , where  $N(\lambda)$  is the norm (product of parts).

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$$\sum_{\delta|\lambda} \mu_{\mathcal{P}}(\delta) = \begin{cases} 1 & \text{if } \lambda = \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

## Parallel universe



### Partition Möbius inversion

If we have

$$f(\lambda) = \sum_{\delta|\lambda} g(\delta)$$

we also have

$$g(\lambda) = \sum_{\delta|\lambda} \mu_{\mathcal{P}}(\lambda/\delta) f(\delta).$$

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- Replacing  $\mathcal{P}$  with  $\mathcal{P}_{\mathbb{P}}$  reduces to classical cases.
- Many analogs of objects in multiplic. # theory...

### Partition Cauchy product

$$\left( \sum_{\lambda \in \mathcal{P}} f(\lambda) q^{|\lambda|} \right) \left( \sum_{\lambda \in \mathcal{P}} g(\lambda) q^{|\lambda|} \right) = \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \sum_{\delta | \lambda} f(\delta) g(\lambda/\delta)$$

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Other multiplicative objects generalize to partition theory...

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- For  $1 \notin \mathcal{P}' = \mathcal{P}_{\mathbb{X}}$  (parts in  $\mathbb{X} \subset \mathbb{N}$ )  $\rightarrow$  Euler product:

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## Partition zeta functions: nice identities

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Summing over partitions into prime parts:

$$\zeta_{\mathcal{P}_{\mathbb{P}}}(\mathbf{2}) = \zeta(\mathbf{2}) = \frac{\pi^2}{6}$$

Summing over partitions into even parts:

$$\zeta_{\mathcal{P}_{\text{even}}}(\mathbf{2}) = \frac{\pi}{2}$$

Summing over partitions into distinct parts:

$$\zeta_{\mathcal{P}_{\text{distinct}}}(\mathbf{2}) = \frac{\sinh \pi}{\pi}$$

### Partition analogs of classical identities

$$\sum_{\lambda \in \mathcal{P}_X} \mu_{\mathcal{P}}(\lambda) N(\lambda)^{-s} = \frac{1}{\zeta_{\mathcal{P}_X}(s)}$$

$$\sum_{\lambda \in \mathcal{P}_X} \varphi_{\mathcal{P}}(\lambda) N(\lambda)^{-s} = \frac{\zeta_{\mathcal{P}_X}(s-1)}{\zeta_{\mathcal{P}_X}(s)}$$

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### Takeaway from these examples



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### Takeaway from these examples

- Different subsets of  $\mathcal{P}$  induce very diff. zeta values

## Partition zeta functions: nice identities

### Partition analogs of classical identities

$$\sum_{\lambda \in \mathcal{P}_X} \mu_{\mathcal{P}}(\lambda) N(\lambda)^{-s} = \frac{1}{\zeta_{\mathcal{P}_X}(s)}$$

$$\sum_{\lambda \in \mathcal{P}_X} \varphi_{\mathcal{P}}(\lambda) N(\lambda)^{-s} = \frac{\zeta_{\mathcal{P}_X}(s-1)}{\zeta_{\mathcal{P}_X}(s)}$$

### Takeaway from these examples

- Different subsets of  $\mathcal{P}$  induce very diff. zeta values
- Classical zeta theorems  $\rightarrow$  partition zeta theorems

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Do there exist (non-Riemann) partition zeta functions such that, for the “right” choice of  $\mathcal{P}' \subsetneq \mathcal{P}$ ,

$$\zeta_{\mathcal{P}'}(N) = \pi^M \times \textit{rational number}?$$

## Partition zeta functions: nice family

Partitions of fixed length  $k$



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Define sum over partitions of fixed length  $\ell(\lambda) = k$ :

$$\zeta_{\mathcal{P}}(\{\mathbf{s}\}^k) := \sum_{\substack{\lambda \in \mathcal{P} \\ \ell(\lambda) = k}} N(\lambda)^{-\mathbf{s}} \quad (\operatorname{Re}(\mathbf{s}) > 1)$$

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Summing over partitions of fixed length  $k > 0$ :

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For  $N \geq 1$ :  $\zeta_{\mathcal{P}}(\{2N\}^k) = \pi^{2Nk} \times \text{rational number}$

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Some *other* partition zeta fctns. contin. to right half-plane.



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### Natural questions

General  $\zeta_{\mathcal{P}}(\{s\}^k)$ ? Analytic continuation? Poles? Roots?

## Partition zeta functions: analytic properties

### Theorem (S.-Sills, 2020)

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For  $\text{Re}(s) > 1$ :

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*Note:*  $\zeta_{\mathcal{P}}(\{s\}^k)$  **not** zero at roots of  $\zeta(s)$  for  $k > 1$

## MacMahon-partition zeta correspondence

Proof mimics Sills' combinatorial proof (2019) of  
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### MacMahon's partial fraction decomposition

For  $|q| < 1$ :

$$\prod_{n=1}^k (1 - q^n)^{-1} = \sum_{\lambda \vdash k} \frac{(1 - q^2)^{-m_2} \cdots q^{km_k} (1 - q^k)^{-m_k}}{N(\lambda) m_1! m_2! \cdots m_k!}$$

Although here we really want to use...

## MacMahon-partition zeta correspondence

### MacMahon's partial fraction decomposition times $q^k$

$$\begin{aligned} q^k \prod_{n=1}^k (1 - q^n)^{-1} &= \sum_{\ell(\lambda)=k} q^{|\lambda|} \\ &= \sum_{\lambda \vdash k} \frac{q^{m_1} (1 - q)^{-m_1} \cdot q^{2m_2} (1 - q^2)^{-m_2} \dots q^{km_k} (1 - q^k)^{-m_k}}{N(\lambda) m_1! m_2! \dots m_k!} \end{aligned}$$

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- RHS = comb. geom. series over partitions of size  $k$

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### Compare and contrast

$$\zeta_{\mathcal{P}}(\{\mathbf{s}\}^k) = \sum_{\ell(\lambda)=k} N(\lambda)^{-s} = \sum_{\lambda \vdash k} \frac{\zeta(\mathbf{s})^{m_1} \zeta(2s)^{m_2} \dots \zeta(ks)^{m_k}}{N(\lambda) m_1! m_2! \dots m_k!}$$



## MacMahon-partition zeta correspondence (from S-Sills)

Gen fctn component	Analogous zeta fctn component
$q^{ \lambda }$	$N(\lambda)^{-s}$
$\frac{q^j}{1-q^j}$	$\zeta(js)$
$\frac{q^k}{\prod_{j=1}^k (1-q^j)}$	$\zeta_{\mathcal{P}}(\{\mathbf{s}\}^k)$

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Multiplication of terms of either shape  $q^n$  or  $n^{-s}$  generates partitions in exactly the same way:

$$q^{\lambda_1} q^{\lambda_2} q^{\lambda_3} \dots q^{\lambda_r} = q^{|\lambda|} \iff \lambda_1^{-s} \lambda_2^{-s} \lambda_3^{-s} \dots \lambda_r^{-s} = N(\lambda)^{-s}$$

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$\frac{q^k}{\prod_{j=1}^k (1-q^j)}$	$\zeta_{\mathcal{P}}(\{s\}^k)$

The term  $q^{jn}$  in geom series  $\sum_{n=1}^{\infty} q^{jn}$  and resp. term  $n^{-js}$  of  $\zeta(js)$  both encode partition  $(n)^j := (n, n, \dots, n)$  ( $j$  times):

$$\frac{q^j}{1-q^j} = \sum_{n=1}^{\infty} q^{|(n)^j|} \quad \longleftrightarrow \quad \zeta(js) = \sum_{n=1}^{\infty} N((n)^j)^{-s}.$$

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**Geometric series-zeta function duality**

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Correspondence says  $\frac{q^j}{1-q^j}$  and  $\zeta(js)$  are interchangeable

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### Geometric series-zeta function duality

Correspondence says  $\frac{q^j}{1-q^j}$  and  $\zeta(js)$  **are interchangeable** as gen fctns for partitions  $(n, n, \dots, n)$  ( $j$  reps / same part).

# Multiplicative theory of (additive) partitions

## Applications

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### Applications

- New combinatorial bijections
- Computing coefficients of  $q$ -series, mock mod. forms
- Statistical physics,  $q$ -bracket
- Computational chemistry
- Computing arithmetic densities, other limits
- Computing  $\pi$



## Multiplicative theory of (additive) partitions

### Applications

- New combinatorial bijections
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- Statistical physics,  $q$ -bracket
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- Computing arithmetic densities, other limits
- Computing  $\pi$  (quite inefficiently, to boot!)

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The *arithmetic density* of a subset  $S \subseteq \mathbb{Z}^+$  is

$$\lim_{N \rightarrow \infty} \frac{\#\{n \in S \mid n \leq N\}}{N},$$

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### Examples

- Integers  $\equiv r \pmod{t}$  have density  $1/t$
- Square-free integers have density  $6/\pi^2 = 1/\zeta(2)$

## Application: arithmetic densities

### Classical computation



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- Well-known relation between arithmetic density and zeta-type sums

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- Well-known relation between arithmetic density and zeta-type sums
- If a subset  $T \subseteq \mathbb{P}$  has arith. density in  $\mathbb{P}$ , its density is equal to the *Dirichlet density* of  $T$

$$\lim_{s \rightarrow 1} \frac{\sum_{p \in T} p^{-s}}{\sum_{p \in \mathbb{P}} p^{-s}}$$

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- extends work of Alladi (1977), Locus Dawsey (2017)

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### Proofs

$q$ -binomial thm + partition bijection + complex analysis

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- another partition-zeta connection

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For  $d_S$  arith. density of subset  $S \subseteq \mathbb{N}$ :

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- here  $*$  is partition convolution,  $\varphi(n)$  classical phi
- second formula extends work of Wang (2020)

## Other arithmetic-partition correspondences (from Ono-S-Wagner)

<b>Natural number <math>n</math></b>	<b>Partition <math>\lambda</math></b>
Prime factors of $n$	Parts of $\lambda$
Square-free integers	Partitions into distinct parts
$\mu(n)$	$\mu_{\mathcal{P}}(\lambda)$
$\varphi(n)$	$\varphi_{\mathcal{P}}(\lambda)$
$\rho_{\min}(n)$	$\text{sm}(\lambda)$
$\rho_{\max}(n)$	$\text{lg}(\lambda)$
$n^{-s}$	$q^{ \lambda }$
$\zeta(s)^{-1}$	$(q; q)_{\infty}$
$s \rightarrow 1$	$q \rightarrow 1$



## Application: partition-theoretic Abelian theorems

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For  $f(n)$  an arithmetic fctn,  $L_f := \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{1 \leq k \leq N} f(k)$ :

$$\lim_{q \rightarrow 1} (q; q)_{\infty} \sum_{\lambda \in \mathcal{P}} f(\lg(\lambda)) q^{|\lambda|} = L_f.$$

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- This is a partition-theoretic analogue of Abel's convergence theorem:

$$\lim_{q \rightarrow 1} (1 - q) \sum_{n \geq 1} f(n) q^n = L_f.$$

## Application: partition-theoretic Abelian theorems

### Corollary (S, 2021)

For  $f(n)$  an arithmetic fctn,  $L_f := \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{1 \leq k \leq N} f(k)$ :

$$\begin{aligned} - \lim_{q \rightarrow 1} \sum_{\lambda \in \mathcal{P}} \mu_{\mathcal{P}}(\lambda) f(\text{sm}(\lambda)) q^{|\lambda|} &= \lim_{q \rightarrow 1} (q; q)_{\infty} \sum_{n \geq 1} \frac{f(n) q^n}{(q; q)_n} \\ &= - \lim_{q \rightarrow 1} \sum_{n \geq 1} \sum_{k \geq 1} \frac{(-1)^k f(n) q^{nk + \frac{k(k-1)}{2}}}{(q; q)_{k-1}} = L_f. \end{aligned}$$

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### Proofs

Abel's convergence thm + manipulations +  $q$ -binomial thm

### Philosophy of this talk (again)

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- Exist multipl., division, arith. functions on partitions
- Objects in classical multiplic. number theory  
→ special cases of partition-theoretic structures
- Expect arithmetic theorems → extend to partitions
- Expect partition properties → properties of integers

### Philosophy of this talk (again)

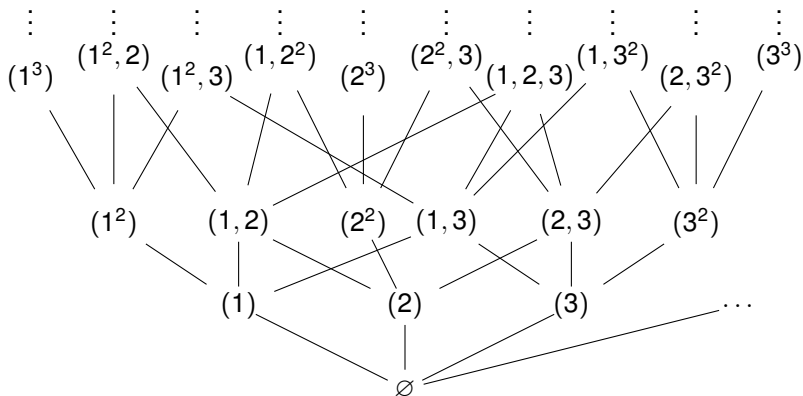
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### Work in progress:

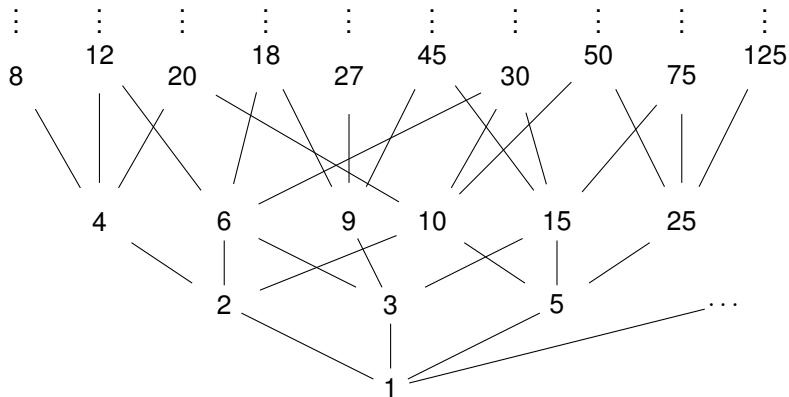
Akande, Dawsey, Hendon, Jameson, Just, Ono, Pulagam, Rolen, M. Schneider, Sellers, Sills, Wagner, students,...



## Partition-theoretic multiverse (diagram from Dawsey-Just-S, 2021)



## Partition-theoretic multiverse (diagram from Dawsey-Just-S, 2021)



## Gratitude

Thank you for listening :)