Composition-theoretic series in partition theory (2)

Drew Sills (Georgia Southern)
Robert Schneider (Michigan Tech)

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$$\prod_{n \in S} (1 - zq^n)^{-1} \; = \; \sum_{\lambda \in \mathcal{P}_S} z^{\ell(\lambda)} q^{|\lambda|},$$

with the sum taken over partitions whose parts all lie in S.

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Our k-gonal number formulas arise from analogous generating function-type identities for compositions.

Composition-theoretic generating functions

For $z, q \in \mathbb{C}$, define an auxiliary series analogous to $(zq; q)_\infty$:

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which converges when $|q| < 1$; note $\phi_S(0; q) = 1$ identically.

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For $S \subseteq \mathbb{N}$, $z, q \in \mathbb{C}$ such that $|q| < \frac{1}{1+|z|}$, we have

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- Restriction on $|q|$ sufficient for convergence for all z, S.
- For arbitrary z, S, necessary condition $|\sum_{n\in S} q^n| < |z^{-1}|$.

Proof based on geometric series

Begin with the multinomial theorem rewritten as a sum over compositions c w/ *largest part* $\lg(c) \leq k$, $\lg(\emptyset) := 0$, $\ell(c) = r$:

$$(x_1 + x_2 + x_3 + \cdots + x_k)^r = \sum_{\substack{0 \leq \lg(c) \leq k \\ \ell(c) = r}} x_1^{m_1} x_2^{m_2} x_3^{m_3} \cdots x_k^{m_k}.$$

Let $k \to \infty$, assuming $|x_1 + x_2 + \ldots| < 1$, and sum over $r \geq 0$:

$$\frac{1}{1 - x_1 - x_2 - \ldots} = \sum_{r \geq 0} (x_1 + x_2 + \ldots)^r = \sum_{c \in \mathcal{C}} x_1^{m_1} x_2^{m_2} x_3^{m_3} \cdots.$$

Set $x_i = zq^i$ if $i \in S$, $x_i = 0$ else, so $x_1 + x_2 + \ldots = 1 - \phi_S(z; q)$:

$$\frac{1}{\phi_S(z; q)} = \sum_{c \in \mathcal{C}_S} (zq)^{m_1} (zq^2)^{m_2} (zq^3)^{m_3} \cdots = \sum_{c \in \mathcal{C}_S} z^{\ell(c)} q^{|c|}.$$

Geom series proof valid when $|1 - \phi_S(z; q)| < 1$; by the triangle inequality we see $|q| < (1 + |z|)^{-1}$ is sufficient (thanks C. Piret).

Translating between partitions and compositions

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The multinomial theorem gives an explicit connection: for each partition $\lambda \in \mathcal{P}$, there are

$$\frac{\ell(\lambda)!}{m_1(\lambda)!\; m_2(\lambda)!\; m_3(\lambda)! \cdots} \;\geq\; 1$$

distinct compositions having the same parts as λ.

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If $g\colon \mathcal{P} \to \mathbb{C}$ is a function symmetric on the parts of $\lambda \in \mathcal{P}$ (i.e., unaffected by order) and $\sum_{\lambda\in\mathcal{P}} g(\lambda)$ converges absolutely, then

$$\sum_{\lambda\in\mathcal{P}} g(\lambda) \;=\; \sum_{c\in\mathcal{C}} \widehat{g}(c),$$

where $\widehat{g}(c) := g(\lambda) \cdot \frac{m_1(\lambda)!m_2(\lambda)!\cdots m_n(\lambda)!}{\ell(\lambda)!}$ for each $c \in \mathcal{C}$ having the same multiset of parts as $\lambda \in \mathcal{P}$.

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Note: Equal sums not necessarily equally "natural" or useful.

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Partition generating functions are connected to modular forms.

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Recall Ramanujan's two theta functions:

$$\psi(q) := \sum_{n\geq 0} q^{n(n+1)/2} = \prod_{k=1}^{\infty}(1-q^k)(1+q^k)^2,$$

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = \prod_{k=1}^{\infty}(1-q^{2k})(1+q^{2k-1})^2.$$

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- Up to mult factors in $q = e^{2\pi i \tau}$, modular of wt $1/2$ in $\tau \in \mathbb{H}$.

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- Up to mult factors in $q = e^{2\pi i \tau}$, modular of wt $1/2$ in $\tau \in \mathbb{H}$.
- Series can be written in the form $\phi_S(z;q) = 1 - z\sum_{n\in S} q^n$:

$$\psi(q) = 1 + \sum_{n\in\Delta} q^n, \qquad \varphi(q) = 1 + 2\sum_{n\in\Box} q^n.$$

Composition-theoretic theta functions

The reciprocals of these theta functions are q-series summed over compositions whose parts are k-gonal numbers:

$$\frac{1}{\psi(q)} = \sum_{c \in \mathcal{C}_\Delta} (-1)^{\ell(c)} q^{|c|}, \qquad \frac{1}{\varphi(q)} = \sum_{c \in \mathcal{C}_\square} (-2)^{\ell(c)} q^{|c|},$$

with $|q| < 1/2$ in the first case, and $|q| < 1/3$ in the second.

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Question: Deeper connections w/ compositions, k-gons, mf's?
Coming soon: W. Keith can prove the results *combinatorially*.

Multiplicative parallels

Define the composition *norm* $N(c)$ by $N(\emptyset) := 1$, and for composition $c = (c_1, c_2, \ldots, c_r)$, by the product of the parts:

$$N(c) := c_1 c_2 c_3 \cdots c_r.$$

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Let $h : \mathcal{C} \to \mathbb{C}$ be a composition-theoretic function. For $\mathcal{C}' \subseteq \mathcal{C}$, $s \in \mathbb{C}$, define composition-theoretic Dirichlet series:

$$\sum_{c \in \mathcal{C}'} h(c) N(c)^{-s}.$$

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Dirichlet series composition-theoretic generating function

For $T \subseteq \mathbb{N} \backslash \{1\}$, $s, z \in \mathbb{C}$ such that $|\sum_{n \in T} n^{-s}| < |z^{-1}|$,

$$\frac{1}{1 - z \sum_{n \in T} n^{-s}} = \sum_{c \in \mathcal{C}_T} z^{\ell(c)} N(c)^{-s}.$$

Note: Compare to $1/\phi_T(z; q) = \sum_{c \in \mathcal{C}_T} z^{\ell(c)} q^{|c|}$.

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The Möbius function $\mu(n)$ is given by the following sum over compositions with no part equal to 1, having norm equal to n:

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Proof. Set $T = N\setminus\{1\}$, $z = -1$, $\mathrm{Re}(s) > 1$ in Dir. series eqn:

$$\frac{1}{\zeta(s)} = \frac{1}{1 + \sum_{n\geq 2} n^{-s}} = \sum_{c \in \mathcal{C}_{\mathbb{N}\setminus\{1\}}} (-1)^{\ell(c)} N(c)^{-s}.$$

Comparing coefficients of n^{-s} on both sides gives the identity.

Note: Sum is over *norm* equal to n, as opposed to size.

Gratitude

THANK YOU FOR LISTENING!