

Hook pairs

Background

Related work

Main result

Combinatorial
proof

Analytic proof

Conclusion

Pairs of hooks of length two

Proofs of two conjectures of Berkovich and Dhar

Speaker: Rong Chen

Shanghai Normal University

Seminar in Partition Theory,
 q -Series and Related Topics

Partitions and Young diagrams

Hook pairs

Background

Related work

Main result

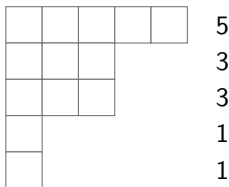
Combinatorial
proof

Analytic proof

Conclusion

A **partition** is a finite sequence of positive integers in weakly decreasing order.

$$\pi = (5, 3, 3, 1, 1), \quad |\pi| = 5 + 3 + 3 + 1 + 1 = 13.$$



The **Young diagram** has one row for each part.

Rows are aligned on the left.

The total number of cells is $|\pi|$.

What is a hook?

Hook pairs

Background

Related work

Main result

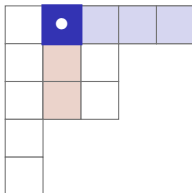
Combinatorial
proof

Analytic proof

Conclusion

The hook of a cell contains:

- the cell itself;
- all cells to its right in the same row;
- all cells below it in the same column.



The **hook length** is the number of cells in the hook.

Here it is

$$1 + 3 + 2 = 6.$$

Hooks of length two

Hook pairs

Background

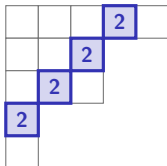
Related work

Main result

Combinatorial
proof

Analytic proof

Conclusion



$$\pi = (5, 3, 3, 1, 1)$$

Four hooks of length two

In an odd partition, each part size greater than 1 gives one horizontal hook of length two. Each repeated part size gives one vertical hook of length two.

A hook of length two has exactly one other cell:

- on the right: **horizontal**;
- below: **vertical**.

Each highlighted cell starts one such hook.

The bounded counting functions

Hook pairs

$$a_2(L, m, n)$$

The number of partitions of n into **odd parts at most $2L - 1$** , with exactly m hooks of length two.

Background

Related work

Main result

Combinatorial proof

Analytic proof

Conclusion

$$b_2(L, m, n)$$

The number of partitions of n into **distinct parts at most L** , with exactly m hooks of length two.

Odd parts are $1, 3, 5, \dots$; repetitions are allowed.

Distinct parts are all different.

Here n is the size, m is the number of hooks, and L sets the bound.

The subscript 2 records the hook length. Both functions count partitions.

Counting pairs of hooks

Hook pairs

A partition with m hooks of length two gives

$$\binom{m}{2} = \frac{m(m-1)}{2}$$

Background

Related work

Main result

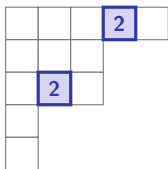
Combinatorial
proof

Analytic proof

Conclusion

unordered pairs of distinct cells with hook length two.

For $(5, 3, 3, 1, 1)$, there are $m = 4$ hooks and $\binom{4}{2} = 6$ pairs.



One selected pair

Thus the total over bounded odd partitions of n is

$$\sum_{m \geq 0} \binom{m}{2} a_2(L, m, n).$$

Equal partition counts, different hook totals

Hook pairs

Theorem 1 (Euler)

For each n , the number of odd partitions of n equals the number of distinct partitions of n .

What happens when each partition is weighted by its number of hooks?

Theorem 2 (Ballantine, Burson, Craig, Folsom, and Wen)

For every $n \geq 0$, the total number of hooks of length two in odd partitions of n is at least the total in distinct partitions of n .

These comparisons have **no bounds on the largest parts**.

Ballantine et al. (2023).

A finite theorem of Berkovich and Dhar

Hook pairs

Using a variant of Sylvester's map, Berkovich and Dhar proved a bounded version of the single-hook comparison.

Background

Related work

Main result

Combinatorial proof

Analytic proof

Conclusion

Theorem 3 (Berkovich and Dhar)

For $L \geq 1$ and $n \geq 0$,

$$\sum_{m \geq 0} m a_2(L, m, n) \geq \sum_{m \geq 0} m b_2(L, m, n).$$

The odd parts are at most $2L - 1$; the distinct parts are at most L .

The weight m counts **one selected hook** in each partition.

For **pairs of hooks**, they proved the unbounded comparison and found generating functions for the bounded families.

Berkovich–Dhar (2025).

Hook pairs: conjecture and Theorem 4

Hook pairs

For $L \geq 1$ and $n \geq 0$:

Conjecture 1 (Berkovich and Dhar)

$$\sum_{m \geq 0} \binom{m}{2} a_2(L, m, n) \geq \sum_{m \geq 0} \binom{m}{2} b_2(L, m, n).$$

Theorem 4

$$\sum_{m \geq 0} \binom{m}{2} a_2(L, m, n) \geq \sum_{m \geq 0} \binom{m}{2} b_2(2L, m, n).$$

The distinct bound increases from L to $2L$. The odd bound stays $2L - 1$.

Theorem 4 proves Conjecture 1. We construct a map between partitions with selected hook pairs.

Conjecture: Berkovich–Dhar (2025).

Conjecture 2 and Theorem 5

Hook pairs

Conjecture 2 (Berkovich and Dhar)

$$\frac{1}{(q; q^2)_L} \left(\frac{1 - q^{2L-2}}{1 - q^2} + q^3 \begin{bmatrix} L-2 \\ 2 \end{bmatrix}_{q^2} \right) \succeq q \begin{bmatrix} L-2 \\ 2 \end{bmatrix}_q (-q^3; q)_{L-4}.$$

Background

Related work

Main result

Combinatorial
proof

Analytic proof

Conclusion

Theorem 5

$$\frac{1}{(q; q^2)_{L-1}} \left(\frac{1 - q^{2L-2}}{1 - q^2} + q^3 \begin{bmatrix} L-1 \\ 2 \end{bmatrix}_{q^2} \right) \succeq q \begin{bmatrix} 2L-2 \\ 2 \end{bmatrix}_q (-q^3; q)_{2L-4}.$$

$\succeq$ denotes coefficientwise inequality.

Ranges: $L \geq 4$ in Conjecture 2; $L \geq 2$ in Theorem 5.

The four sets: marked partitions

Hook pairs

Fix $L \geq 2$ and $n \geq 0$. In each marked partition, S is a set of two distinct cells of hook length two.

$$\mathcal{D} = \left\{ (\pi_d, S) : \begin{array}{l} \pi_d \text{ has distinct parts at most } 2L, \\ |\pi_d| = n \end{array} \right\},$$

$$\mathcal{O} = \left\{ (\rho, S) : \begin{array}{l} \rho \text{ has odd parts at most } 2L - 1, \\ |\rho| = n \end{array} \right\}.$$

For a partition with m two-hooks, there are $\binom{m}{2}$ choices of S .

$$\#\mathcal{D} = \sum_{m \geq 0} \binom{m}{2} b_2(2L, m, n), \quad \#\mathcal{O} = \sum_{m \geq 0} \binom{m}{2} a_2(L, m, n).$$

The three maps connect these sets through $\mathcal{T}$ and $\mathcal{U}$:

$$\mathcal{D} \xrightarrow{\alpha} \mathcal{T} \xrightarrow{\phi} \mathcal{U} \xrightarrow{\psi} \mathcal{O}.$$

The second set: right side of Theorem 5

Hook pairs

The second set consists of the following triples:

$$\mathcal{T} = \left\{ (\mu, a, b) : \begin{array}{l} \mu \text{ has distinct parts in } \{3, \dots, 2L-2\}, \\ a, b \in \mathbb{Z}, \quad 0 \leq a \leq b \leq 2L-4, \\ |\mu| + 1 + a + b = n - 5 \end{array} \right\}.$$

The **weight** of (μ, a, b) is $|\mu| + 1 + a + b$. Thus every element of $\mathcal{T}$ has weight $n - 5$.

The **right side of Theorem 5** is

$$q^{\left[\begin{smallmatrix} 2L-2 \\ 2 \end{smallmatrix} \right]}_q (-q^3; q)_{2L-4}.$$

The factors account for the contributions 1, $a + b$, and $|\mu|$ to the weight.

Its coefficient of q^{n-5} is $\#\mathcal{T}$.

Background

Related work

Main result

Combinatorial
proof

Analytic proof

Conclusion

The third set: left side of Theorem 5

Hook pairs

The third set is

$$\mathcal{U} = \left\{ (\pi, k) : \begin{array}{l} \pi \text{ has odd parts at most } 2L-3, \\ k \in \mathbb{Z}, \quad 0 \leq k \leq L-2, \\ |\pi| + 2k = n-5 \end{array} \right\}$$

$$\cup \left\{ (\pi, c, d) : \begin{array}{l} \pi \text{ has odd parts at most } 2L-3, \\ c, d \in \mathbb{Z}, \quad 0 \leq c \leq d \leq L-3, \\ |\pi| + 3 + 2c + 2d = n-5 \end{array} \right\}.$$

The two weights are $|\pi| + 2k$ and $|\pi| + 3 + 2c + 2d$. Both equal $n-5$ on $\mathcal{U}$.

The **left side of Theorem 5** is

$$\frac{1}{(q; q^2)_{L-1}} \left(\frac{1 - q^{2L-2}}{1 - q^2} + q^3 \begin{bmatrix} L-1 \\ 2 \end{bmatrix}_{q^2} \right).$$

Its coefficient of q^{n-5} is $\#\mathcal{U}$. The map $\phi : \mathcal{T} \rightarrow \mathcal{U}$ preserves weight.

For $L = 2$, the second set in this union is empty.

One fixed element of $\mathcal{D}$

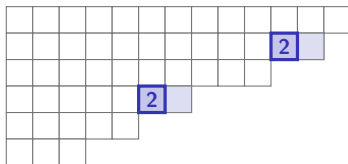
Hook pairs

Fix $L = 7$, $n = 50$, and

$$\pi_d = (13, 12, 10, 7, 5, 3).$$

$i = 2$

$j = 4$



The two highlighted starting cells form S .

We follow the element

$$(\pi_d, S) \in \mathcal{D}.$$

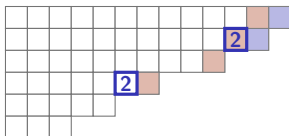
The row indices i, j are counted from the top.

This partition has $m = 5$ hooks of length two, so it contributes $\binom{5}{2} = 10$ elements to $\mathcal{D}$. We have fixed one of them.

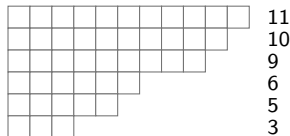
α : shorten the first rows

Hook pairs

For $i = 2, j = 4$, remove one cell from each of the first 2 rows, then one from each of the first 4 rows.



π_d , size 50



$\delta = (11, 10, 9, 6, 5, 3)$

Blue: first i rows. Orange: first j rows.

This step gives the triple

$$(\delta, i, j) = ((11, 10, 9, 6, 5, 3), 2, 4), \quad |\delta| = 44.$$

General rule. Shorten the first i and first j rows by one cell each. The marked hooks ensure that δ remains distinct and positive, with $|\delta| = n - i - j$.

α : select two parts of δ

Hook pairs

Background

Related work

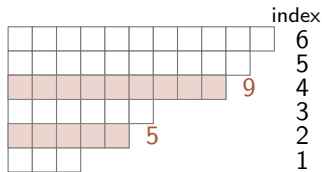
Main result

Combinatorial
proof

Analytic proof

Conclusion

For $(\delta, i, j) = (\delta, 2, 4)$, select the i -th and j -th parts **from the bottom** of δ .



The 2nd smallest part is 5; the 4th smallest is 9.

$$a = 5 - 1 = 4, \quad b = 9 - 2 = 7.$$

Thus the selected parts are $a + 1$ and $b + 2$.

General rule. Let $\delta = (\delta_1, \dots, \delta_r)$, with parts indexed from top to bottom. Define

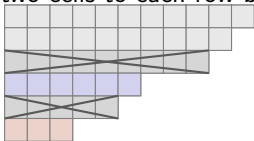
$$a = \delta_{r-i+1} - 1, \quad b = \delta_{r-j+1} - 2.$$

This gives the triple $(\delta, a, b) = (\delta, 4, 7)$.

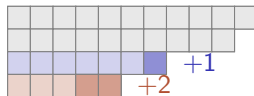
α : delete two rows and adjust the rest

Hook pairs

Delete the parts 5 and 9. Add one cell to each row between them and two cells to each row below both.



→



$$\mu = (11, 10, 7, 5)$$

$$\delta = (11, 10, 9, 6, 5, 3)$$

These steps define $\alpha : \mathcal{D} \rightarrow \mathcal{T}$. In our example,

$$\alpha(\pi_d, S) = (\mu, a, b) = ((11, 10, 7, 5), 4, 7) \in \mathcal{T}.$$

$$|\mu| + 1 + a + b = 33 + 1 + 4 + 7 = 45 = n - 5.$$

ϕ : the case $a > 0$

Hook pairs

Background

Related work

Main result

Combinatorial
proof

Analytic proof

Conclusion

We now have the element

$$(\mu, a, b) = ((11, 10, 7, 5), 4, 7) \in \mathcal{T}.$$

The odd numbers $a - 1 = 3$ and $b - a = 3$ give $f = e = 1$. Then

$$4 = 2 \cdot 1 + 1 + 1, \quad 7 = 2 \cdot 2 + 1 + 1 + 1.$$

Thus $c = 1$, $d = 2$ and $e = f = 1$.

General rule. The indices c, d, e, f are uniquely determined by

$$a = 2c + 1 + f, \quad b = 2d + 1 + f + e, \quad e, f \in \{0, 1\}.$$

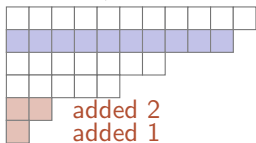
Adjoining e parts equal to 1 and f parts equal to 2 gives the triple

$$(\mu \cup (1^e, 2^f), c, d), \quad 0 \leq c \leq d \leq L - 3.$$

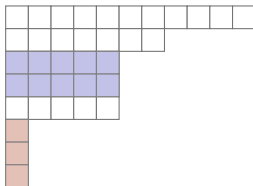
ϕ : apply Glaisher's bijection

Hook pairs

Apply **Glaisher's bijection** g to the partition in this triple: a part $2^s u$, with u odd, becomes 2^s copies of u . Here 10 splits into $5 + 5$, and 2 into $1 + 1$.



$\xrightarrow{g}$



$$\begin{aligned} &\mu \cup (1, 2) \\ &= (11, 10, 7, 5, 2, 1) \end{aligned}$$

$$\pi = (11, 7, 5^3, 1^3)$$

These steps give the image under ϕ . In our example,

$$\phi(\mu, a, b) = (\pi, c, d) = ((11, 7, 5^3, 1^3), 1, 2) \in \mathcal{U}.$$

$$|\pi| + 3 + 2c + 2d = 36 + 3 + 2 + 4 = 45.$$

The bijection g preserves size.

ϕ : the complete rule

Hook pairs

For $a > 0$, the preceding steps give

$$\phi(\mu, a, b) = (g(\mu \cup (1^e, 2^f)), c, d) \in \mathcal{U}.$$

The resulting partition has odd parts at most $2L - 3$.

For $a = 0$, the image is a pair in $\mathcal{U}$:

$$\phi(\mu, 0, 2k) = (g(\mu \cup (1)), k),$$

$$\phi(\mu, 0, 2k + 1) = (g(\mu \cup (2)), k).$$

Together, these rules define $\phi : \mathcal{T} \rightarrow \mathcal{U}$. Every image has weight $n - 5$.

Background

Related work

Main result

Combinatorial
proof

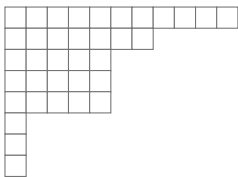
Analytic proof

Conclusion

ψ : use c, d to create the marked hooks

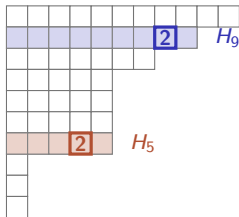
Hook pairs

Our element of $\mathcal{U}$ is $((11, 7, 5^3, 1^3), 1, 2)$. Add $2c + 3 = 5$ and $2d + 5 = 9$.



$$\pi = (11, 7, 5^3, 1^3)$$

$\rightarrow$



$$\rho = (11, 9, 7, 5^4, 1^3)$$

For each inserted size, mark the second-last cell of its bottommost row. These cells are $H_5(\rho)$ and $H_9(\rho)$.

$$\psi(\pi, c, d) = (\rho, S) = (\rho, \{H_5(\rho), H_9(\rho)\}) \in \mathcal{O}.$$

$$|\rho| = 36 + 5 + 9 = 50.$$

ψ : the complete rule

Hook pairs

For an odd partition ρ , use the following cells when they exist. $H_s(\rho)$ is the second-last cell in the bottommost row of length $s > 1$; $V_1(\rho)$ is the cell in the second-bottommost row of length 1.

The type used in our example, (π, c, d) :

$$\rho = \pi \cup (2c + 3, 2d + 5), \quad S = \{H_{2c+3}(\rho), H_{2d+5}(\rho)\}.$$

The other type, (π, k) :

$$\rho = \pi \cup (1^2, 2k + 3), \quad S = \{V_1(\rho), H_{2k+3}(\rho)\}.$$



The resulting element is $(\rho, S) \in \mathcal{O}$: $|\rho| = n$, all parts are at most $2L - 1$, and both marked cells have hook length two.

These rules define $\psi : \mathcal{U} \rightarrow \mathcal{O}$.

Theorem 4 from the four sets

Hook pairs

We have constructed

$$\mathcal{D} \xrightarrow{\alpha} \mathcal{T} \xrightarrow{\phi} \mathcal{U} \xrightarrow{\psi} \mathcal{O}.$$

The corresponding counting inequalities are

$$\#\mathcal{D} \leq \#\mathcal{T} \leq \#\mathcal{U} \leq \#\mathcal{O}.$$

The middle inequality is Theorem 5. Hence we obtain:

Theorem 4

$$\sum_{m \geq 0} \binom{m}{2} b_2(2L, m, n) \leq \sum_{m \geq 0} \binom{m}{2} a_2(L, m, n).$$

In our example, π_d has 5 two-hooks and ρ has 6. Each element records **one chosen pair**.

$$50 \xrightarrow{\alpha} 45 \xrightarrow{\phi} 45 \xrightarrow{\psi} 50 \quad (\text{size, weight, weight, size}).$$

Background

Related work

Main result

Combinatorial
proof

Analytic proof

Conclusion

Generating functions for hook pairs

Hook pairs

Distinct parts at most $2L$:

$$\sum_{n,m \geq 0} \binom{m}{2} b_2(2L, m, n) q^n = q^6 \begin{bmatrix} 2L-2 \\ 2 \end{bmatrix}_q (-q^3; q)_{2L-4}.$$

Odd parts at most $2L-1$:

$$\begin{aligned} & \sum_{n,m \geq 0} \binom{m}{2} a_2(L, m, n) q^n \\ &= \frac{1}{(q; q^2)_L} \left(q^5 \frac{1 - q^{2L-2}}{1 - q^2} + q^8 \begin{bmatrix} L-1 \\ 2 \end{bmatrix}_{q^2} \right. \\ & \quad \left. + (q^6 + q^8) \frac{1 - q^{4L-4}}{1 - q^4} + q^{16} \begin{bmatrix} L-1 \\ 2 \end{bmatrix}_{q^4} \right. \\ & \quad \left. + \sum_{2 \leq i < j \leq L} (q^{2i+4j-3} + q^{4i+2j-3}) \right). \end{aligned}$$

The lower bound and Theorem 5

Hook pairs

All terms have nonnegative coefficients. Thus

$$\begin{aligned} \sum_{n,m \geq 0} \binom{m}{2} a_2(L, m, n) q^n \\ \succeq \frac{q^5}{(q; q^2)_L} \left(\frac{1 - q^{2L-2}}{1 - q^2} + q^3 \begin{bmatrix} L-1 \\ 2 \end{bmatrix}_{q^2} \right) \\ \succeq \frac{q^5}{(q; q^2)_{L-1}} \left(\frac{1 - q^{2L-2}}{1 - q^2} + q^3 \begin{bmatrix} L-1 \\ 2 \end{bmatrix}_{q^2} \right). \end{aligned}$$

The second inequality uses $(1 - q^{2L-1})^{-1} \succeq 1$.

The last expression is q^5 times the **left side of Theorem 5**. The distinct-part generating function is q^5 times its right side.

A finite product comparison

Hook pairs

Cancellation gives the first key comparison:

$$\begin{aligned}\frac{1}{(q; q^2)_{L-1}} &= \frac{(-q; q)_{2L-2}}{(q^{2L}; q^2)_{L-1}} \\ &\succeq (1+q)(1+q^2)(-q^3; q)_{2L-4}.\end{aligned}$$

Here $(q^{2L}; q^2)_{L-1}^{-1}$ has nonnegative coefficients and constant term 1.

Since $(-q^3; q)_{2L-4}$ has nonnegative coefficients, it suffices to prove the finite comparison:

$$(1+q)(1+q^2) \left(\frac{1-q^{2L-2}}{1-q^2} + q^3 \begin{bmatrix} L-1 \\ 2 \end{bmatrix}_{q^2} \right) \succeq q \begin{bmatrix} 2L-2 \\ 2 \end{bmatrix}_q.$$

Background

Related work

Main result

Combinatorial
proof

Analytic proof

Conclusion

The key finite identity

Hook pairs

The difference in the preceding comparison has a simple form:

$$\begin{aligned} & (1+q)(1+q^2) \left(\frac{1-q^{2L-2}}{1-q^2} + q^3 \begin{bmatrix} L-1 \\ 2 \end{bmatrix}_{q^2} \right) \\ & \quad - q \begin{bmatrix} 2L-2 \\ 2 \end{bmatrix}_q \\ & = \frac{1-q^{2L-2}}{1-q^2} (1+q^3+q^{2L-2}) \succeq 0. \end{aligned}$$

Indeed,

$$\frac{1-q^{2L-2}}{1-q^2} = 1 + q^2 + \dots + q^{2L-4}.$$

Multiplication by $(-q^3; q)_{2L-4}$ and the product comparison prove

Theorem 5.

The analytic comparison

Hook pairs

Background

Related work

Main result

Combinatorial
proof

Analytic proof

Conclusion

$$\begin{aligned} & \sum_{n,m \geq 0} \binom{m}{2} a_2(L, m, n) q^n \\ & \succeq \frac{q^5}{(q; q^2)_{L-1}} \left(\frac{1 - q^{2L-2}}{1 - q^2} + q^3 \begin{bmatrix} L-1 \\ 2 \end{bmatrix}_{q^2} \right) \\ & \stackrel{\text{Theorem 5}}{\succeq} q^6 \begin{bmatrix} 2L-2 \\ 2 \end{bmatrix}_q (-q^3; q)_{2L-4} = \sum_{n,m \geq 0} \binom{m}{2} b_2(2L, m, n) q^n. \end{aligned}$$

Comparing the coefficients of q^n proves Theorem 4:

$$\sum_{m \geq 0} \binom{m}{2} a_2(L, m, n) \geq \sum_{m \geq 0} \binom{m}{2} b_2(2L, m, n).$$

For $L = 1$, both hook-pair counts are zero.

Hook pairs

Background

Related work

Main result

Combinatorial
proof

Analytic proof

Conclusion

Thank you!

arXiv:2609.26324