

# Linear identities for partition pairs with 4-cores and 5-cores

Russelle Guadalupe  
Institute of Mathematics, University of the Philippines Diliman

12 February 2026  
Seminar in Partition Theory,  $q$ -Series and Related Topics  
Michigan Technological University



A **partition** of a given  $n \in \mathbb{N}$  is a nonincreasing sequence  $(\lambda_1, \dots, \lambda_k)$  of positive integers whose sum is  $n$ . The numbers  $\lambda_k$  are referred to as the **parts** of  $n$ .

For example, there are five partitions of 4, namely

$$4,$$

$$3 + 1,$$

$$2 + 2,$$

$$2 + 1 + 1,$$

$$1 + 1 + 1 + 1.$$

Let  $p(n)$  count the number of partitions of  $n$ .

- Euler (1748): We have that

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{f_1},$$

where  $f_k := \prod_{n=1}^{\infty} (1 - q^{kn})$  for  $q \in \mathbb{C}$  with  $|q| < 1$ .

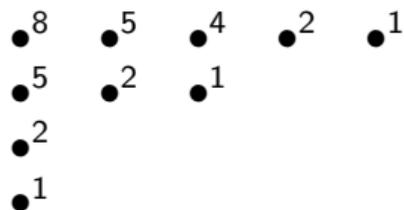
- Ramanujan (1919), Watson (1938), Atkin (1967): For  $\ell \in \{5, 7, 11\}$ , we have

$$p(n) \equiv 0 \pmod{\ell^\beta}$$

for all  $\alpha \geq 1$  and  $n \geq 1$  such that  $24n \equiv 1 \pmod{\ell^\beta}$ , where  $\beta = \alpha$  if  $\ell \in \{5, 11\}$  and  $\beta := \lfloor \alpha/2 \rfloor + 1$  if  $\ell = 7$ .

We can visualize a given partition  $(\lambda_1, \dots, \lambda_k)$  of  $n$  via its Ferris–Young diagram, which is described as follows: We arrange the nodes in  $k$  left-aligned rows so that the  $j$ th row has  $\lambda_j$  nodes. We then assign to a node at a point  $(i, j)$  its hook number, which is the total number of dots directly below and to the right of that node, including the node itself.

For example, for the partition  $(5, 3, 1, 1)$  of 10, we find its Ferris–Young diagram and the corresponding hook numbers (indicated by superscripts) as shown:



We say that a partition of  $n$  is  $t$ -**core** for some  $t \in \mathbb{N}$  if none of its hook numbers is divisible by  $t$ . Thus, the partition  $(5, 3, 1, 1)$  of 10 is  $t$ -core for  $t \in \{3, 6, 7\}$  and  $t \geq 9$ .

Let  $c_t(n)$  be the number of partitions of  $n$  that are  $t$ -cores. Garvan, Kim, and Stanton (1990) found via combinatorial arguments that

$$\sum_{n=0}^{\infty} c_t(n)q^n = \frac{f_t^t}{f_1}.$$

Various authors have studied arithmetic properties for  $c_t(n)$ :

- Garvan (1990): congruences for  $c_p(n)$  for certain primes  $p$  via modular forms
- Hirschhorn and Sellers (1996): infinite families of linear identities for  $c_4(n)$  given by

$$c_4 \left( 3^{2k+1}n + \frac{5 \cdot 3^{2k} - 5}{8} \right) = 3^k c_4(3n),$$

$$c_4 \left( 3^{2k+1}n + \frac{13 \cdot 3^{2k} - 5}{8} \right) = (2 \cdot 3^k - 1) c_4(3n + 1),$$

$$c_4 \left( 3^{2k+2}n + \frac{7 \cdot 3^{2k+1} - 5}{8} \right) = \frac{3^{k+1} - 1}{2} c_4(9n + 2),$$

$$c_4 \left( 3^{2k+2}n + \frac{23 \cdot 3^{2k+1} - 5}{8} \right) = \frac{3^{k+1} - 1}{2} c_4(9n + 8)$$

for all  $n \geq 0$  and  $k \geq 1$

- Ono and Sze (1997): a formula for  $c_4(n)$  in terms of a certain class number

- Baruah and Berndt (2007): linear identities for  $c_3(n)$  and  $c_5(n)$  via Ramanujan's modular equations, namely

$$c_3(4n + 1) = c_3(n),$$

$$c_5(4n + 3) = c_5(2n + 1) + 2c_5(n)$$

for all  $n \geq 0$

- Kim (2010): linear identities for  $c_p(n)$  for primes  $p \geq 5$  via Hecke operators
- Baruah and Nath (2013): new infinite families of linear identities for  $c_4(n)$
- Park (2014): proofs of linear identities for  $c_3(n)$  and  $c_5(n)$  via modular function theory
- G (2026+): proof of a linear identity for  $c_5(n)$  via Ramanujan's function  $k(q)$

We define a partition pair of  $n$  with  $t$ -cores by a pair of partitions  $(\lambda, \mu)$  such that the sum of all parts of  $\lambda$  and  $\mu$  is  $n$  and both  $\lambda$  and  $\mu$  are  $t$ -cores. Then the generating function for the number  $A_t(n)$  of partition pairs of  $n$  with  $t$ -cores is given by

$$\sum_{n=0}^{\infty} A_t(n)q^n := \frac{f_t^{2t}}{f_1^2}.$$

Several authors have explored arithmetic properties for  $A_t(n)$ :

- Lin (2014), Xia (2015), Yao (2015): congruences for  $A_3(n)$
- Baruah and Nath (2015): infinite families of linear identities for  $A_3(n)$  such as

$$A_3\left(2^{2k+1}n + \frac{5 \cdot 2^{2k} - 2}{3}\right) = (2^{2k+1} - 1)A_3(2n + 1)$$

for  $k \geq 0$  and  $n \geq 0$

- Dasappa (2016): congruences modulo powers of 5 for  $A_5(n)$
- Saikia and Boruah (2017): parity results for  $A_4(n)$

The goal of this work is to describe analogous properties for  $A_4(n)$  and  $A_5(n)$ . In particular, we present infinite families of linear identities for  $A_4(n)$  and  $A_5(n)$ .

### Theorem 1 (G (2026))

For integers  $k \geq 1$ , we have

$$\sum_{n=-1}^{\infty} A_5(2^k n + 2^{k+1} - 2)q^n = B_k \frac{f_1^4 f_5^4}{q} - 8B_{k-1} f_2^4 f_{10}^4 + \frac{8^{k+1} - 1}{7} \cdot \frac{f_5^{10}}{f_1^2} - \frac{8^{k+1} - 8}{7} \cdot \frac{q^2 f_{10}^{10}}{f_2^2}, \quad (1)$$

where the sequence  $\{B_k\}_{k \geq 0}$  is defined by  $B_0 = 0$ ,  $B_1 = 1$ , and

$$B_k = -4B_{k-1} - 8B_{k-2} + \frac{8^k - 1}{7}$$

for  $k \geq 2$ .

**Theorem 2 (G (2026))**

For all integers  $n \geq 0$  and  $k \geq 1$ , we have

$$A_5(2^{k+1}n + 3 \cdot 2^k - 2) = B_k A_5(4n + 4) + \left( \frac{8^{k+1} - 1}{7} - 9B_k \right) A_5(2n + 1), \quad (2)$$

where  $\{B_k\}_{k \geq 0}$  is the sequence defined in Theorem 1. Consequently, for all  $m \geq 0$  and  $n \geq 0$ , we have

$$A_5(2^{4m+4}n + 3 \cdot 2^{4m+3} - 2) \equiv 0 \pmod{\frac{8^{4m+4} - 1}{91}}. \quad (3)$$

### Theorem 3 (G (2026))

For integers  $k \geq 0$ , we have

$$\sum_{n=0}^{\infty} A_4 \left( 3^{2k+1} n + \frac{3^{2k+2} - 5}{4} \right) q^n = \frac{3^{2k+2} - 1}{4} \cdot \frac{f_2^{12}}{f_1^6} + \frac{27(3^{2k} - 1)(3^{2k+2} - 1)}{320} \left( 3 \frac{f_2^2 f_3^{10}}{f_1^4 f_6^2} + 4q \frac{f_3 f_6^7}{f_1 f_2} \right) + 3^{2k+2} \cdot \frac{q^3 f_{12}^8}{f_3^2}. \quad (4)$$

### Theorem 4 (G (2026))

For all integers  $n \geq 0$  with  $3 \nmid n$  and  $k \geq 0$ , we have

$$A_4 \left( 3^{2k+1}n + \frac{3^{2k+2} - 5}{4} \right) = \frac{3^{4k+4} - 1}{80} A_4(3n + 1). \quad (5)$$

Consequently, for all integers  $n \geq 0$  with  $3 \nmid n$  and  $k \geq 0$ , we have

$$A_4 \left( 3^{2k+1}n + \frac{3^{2k+2} - 5}{4} \right) \equiv 0 \pmod{\frac{3(3^{4k+4} - 1)}{40}}. \quad (6)$$

To prove our main results, we rely on rigorous elementary  $q$ -series techniques; more precisely, we employ certain 2- and 3-dissection formulas and some identities involving the Ramanujan's cubic continued fraction

$$C(q) := \frac{1}{1 + \frac{q + q^2}{1 + \frac{q^2 + q^4}{1 + \frac{q^3 + q^6}{1 + \dots}}}} = \frac{f_1 f_6^3}{f_2 f_3^3},$$

the Ramanujan's function

$$k(q) := q \prod_{n=1}^{\infty} \frac{(1 - q^{10n-9})(1 - q^{10n-8})(1 - q^{10n-2})(1 - q^{10n-1})}{(1 - q^{10n-7})(1 - q^{10n-6})(1 - q^{10n-4})(1 - q^{10n-3})}$$

and its level 12 analogue (aka the continued fraction of order 12)

$$\begin{aligned}
 h(q) &:= \frac{q(1-q)}{1-q^3 + \frac{q^3(1-q^2)(1-q^4)}{(1-q^3)(1+q^6) + \frac{q^3(1-q^8)(1-q^{10})}{(1-q^3)(1+q^{12}) + \dots}} \\
 &= q \prod_{n=1}^{\infty} \frac{(1-q^{12n-1})(1-q^{12n-11})}{(1-q^{12n-5})(1-q^{12n-7})}
 \end{aligned}$$

due to Mahadeva Naika, B. N. Dharmendra, and Shivashankara (2008).

We first require the following 2-dissections

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q \frac{f_2^2 f_8^4}{f_4^2}, \quad (7)$$

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q \frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}. \quad (8)$$

Identity (7) follows from replacing  $q$  with  $-q$  in Lemma 2.3 of Hirschhorn and Roselin (2010).  
 Identity (8) is Theorem 2.1 of Hirschhorn and Sellers (2010).

We next require the following identities for  $k(q)$ , which can be found in Theorem 10.4 of Cooper's book (2017):

$$\frac{f_2 f_5^5}{q f_1 f_{10}^5} = \frac{1}{k(q)} - k(q), \quad (9)$$

$$\frac{f_2^4 f_5^2}{q f_1^2 f_{10}^4} = \frac{1}{k(q)} + 1 - k(q), \quad (10)$$

$$\frac{f_1^3 f_5}{q f_2 f_{10}^3} = \frac{1}{k(q)} - 4 - k(q). \quad (11)$$

We also need the following identity, which is Theorem 3.3 of Chern and Tang (2022):

$$\frac{k(q)}{k(q^2)} - \frac{k(q^2)}{k(q)} = \frac{f_1 f_5^3}{q f_{10}^4}. \quad (12)$$

### Lemma 5

We have the identity

$$X^2 - Y + 2XY + X^2Y + Y^2 = 0,$$

where  $X := k(q)$  and  $Y := k(q^2)$ .

*Proof outline:* Square (12) and use (9) and (11) to find that

$$\frac{(X^2 + Y - 2XY - X^2Y + Y^2)(X^2 - Y + 2XY + X^2Y + Y^2)}{X^2Y^2} = 0. \quad (13)$$

Using the  $q$ -expansion of  $k(q)$ , we see that the second factor of the numerator of (13) vanishes while the first factor does not, arriving at the desired identity.  $\square$

### Lemma 6 (G)

We have the identity

$$\frac{f_2^3 f_{10}^9}{f_1^3 f_4 f_5 f_{20}^3} - 4q^2 \frac{f_4 f_5^2 f_{20}^3}{f_1^2} = \frac{f_5^5}{f_1} + 2q \frac{f_{10}^5}{f_2}.$$

### Remark

G (2026+) used the above result to give another elementary proof of the linear identity for  $c_5(n)$ .

*Proof.* We recast the given identity as

$$\frac{f_2^4 f_{10}^4}{q f_1^3 f_4 f_5 f_{20}^3} - 4q \frac{f_2 f_4 f_5^2 f_{20}^3}{f_1^2 f_{10}^5} = \frac{f_2 f_5^5}{q f_1 f_{10}^5} + 2. \quad (14)$$

Let  $X := k(q)$  and  $Y := k(q^2)$ . Replacing  $q$  with  $q^2$  in (11) yields

$$Z := \frac{1}{Y} - 4 - Y = \frac{f_2^3 f_{10}}{q^2 f_4 f_{20}^3}. \quad (15)$$

We know from Lemma 5 and (15) that

$$1 - 2X - X^2 = \frac{X^2}{Y} + Y = X^2(Z + 4 + Y) + Y = X^2(Z + 4) + Y(X^2 + 1),$$

so that

$$Y = \frac{1 - 2X - X^2 - X^2(Z + 4)}{X^2 + 1}. \quad (16)$$

*Proof (continued).* Plugging in (16) into Lemma 5 and clearing denominators, we find that

$$X^2 Z^2 + (-1 - 2X + X^2)(1 - 4X - X^2)Z - 4(1 + X - X^2)(1 - 4X - X^2) = 0. \quad (17)$$

Hence, by (9), (11), (15), and (17), we deduce that

$$\begin{aligned} \frac{f_2^4 f_{10}^4}{q f_1^3 f_4 f_5 f_{20}^3} - 4q \frac{f_2 f_4 f_5^2 f_{20}^3}{f_1^2 f_{10}^5} &= \frac{f_2^3 f_{10}}{q^2 f_4 f_{20}^3} \cdot \frac{q f_2 f_{10}^3}{f_1^3 f_5} - 4 \cdot \frac{q^2 f_4 f_{20}^3}{f_2^3 f_{10}} \cdot \frac{f_2^4 f_5^2}{q f_1^2 f_{10}^4} \\ &= \frac{ZX}{1 - 4X - X^2} - 4 \cdot \frac{1 + X - X^2}{ZX} \\ &= \frac{Z^2 X^2 - 4(1 + X - X^2)(1 - 4X - X^2)}{ZX(1 - 4X - X^2)} \\ &= \frac{(1 + 2X - X^2)(1 - 4X - X^2)Z}{ZX(1 - 4X - X^2)} = \frac{1}{X} - X + 2 = \frac{f_2 f_5^5}{q f_1 f_{10}^5} + 2, \end{aligned}$$

which is exactly (14). □

## Lemma 7

We have the identity

$$\frac{f_4 f_{10}^{12}}{f_1^2 f_5^2 f_{20}^5} + 4q^3 \frac{f_2^3 f_5^3 f_{20}^5}{f_1^3 f_4 f_{10}^3} = \frac{f_2^3 f_5^8}{f_1^4 f_{10}^3} - 2q \frac{f_2^2 f_5^3 f_{10}^2}{f_1^3}.$$

*Proof.* Dividing both sides of the given identity by  $qf_2^2 f_5^3 f_{10}^2 / f_1^3$ , it suffices to prove that

$$\frac{f_1 f_4 f_{10}^{10}}{q f_2^2 f_5^5 f_{20}^5} + 4q^2 \frac{f_2 f_{20}^5}{f_4 f_{10}^5} = \frac{f_2 f_5^5}{q f_1 f_{10}^5} - 2. \quad (18)$$

Let  $X := k(q)$  and  $Y := k(q^2)$ . We replace  $q$  with  $q^2$  in (9) so that

$$W := \frac{1}{Y} - Y = \frac{f_4 f_{10}^5}{q^2 f_2 f_{20}^5}. \quad (19)$$

By Lemma 5 and (19), we have

$$1 - 2X - X^2 = \frac{X^2}{Y} + Y = X^2(W + Y) + Y,$$

so that

$$Y = \frac{1 - 2X - X^2 - X^2 W}{X^2 + 1}. \quad (20)$$

*Proof (continued).* Combining Lemma 5 and (20) and clearing denominators, we see that

$$X^2W^2 - (1 - 2X - X^2)(1 - X^2)W + 4X(1 - X^2) = 0. \quad (21)$$

We infer from (9), (19), and (21) that

$$\begin{aligned} \frac{f_1 f_4 f_{10}^{10}}{q f_2^2 f_5^5 f_{20}^5} + 4q^2 \frac{f_2 f_{20}^5}{f_4 f_{10}^5} &= \frac{WX}{1 - X^2} + \frac{4}{W} = \frac{XW^2 + 4(1 - X^2)}{(1 - X^2)W} \\ &= \frac{(1 - 2X - X^2)(1 - X^2)W}{X(1 - X^2)W} = \frac{1}{X} - 2 - X = \frac{f_2 f_5^5}{q f_1 f_{10}^5} - 2, \end{aligned}$$

which is exactly (18). □

## Lemma 8

We have the identity

$$\frac{f_1^2 f_4^2 f_{10}^2}{q f_2^2 f_5^2 f_{20}^2} - \frac{f_2^4 f_{20}^2}{f_4^2 f_{10}^4} = \frac{f_1^3 f_5}{q f_2 f_{10}^3}.$$

*Proof.* Let  $X := k(q)$  and  $Y := k(q^2)$ . By (9) and (11), we have

$$A := \frac{1 - 4X - X^2}{1 - X^2} = \frac{f_1^4 f_{10}^2}{f_2^2 f_5^4},$$

$$B := \frac{1 - 4Y - Y^2}{1 - Y^2} = \frac{f_2^4 f_{20}^2}{f_4^2 f_{10}^4}.$$

Then from (19) we have

$$\frac{4}{1 - A} = \frac{1}{X} - X, \quad (22)$$

$$W = \frac{4}{1 - B} = \frac{1}{Y} - Y. \quad (23)$$

Dividing both sides of (21) by  $X^2$  and applying (22), we get

$$W^2 - \left( \frac{4}{1 - A} - 2 \right) \frac{4W}{1 - A} + \frac{16}{1 - A} = 0. \quad (24)$$

*Proof (continued).* We now substitute (23) into (24) and clear denominators, yielding

$$A^2 + 4AB + B^2 - 5A - AB^2 = 0. \quad (25)$$

Applying (11), (22), (25), and the definitions of  $A$  and  $B$ , we arrive at

$$\begin{aligned} \frac{f_1^2 f_4^2 f_{10}^2}{q f_2^2 f_5^2 f_{20}^2} - \frac{f_2^4 f_{20}^2}{f_4^2 f_{10}^4} &= \frac{A}{B} \left( \frac{4}{1-A} + 1 \right) - B = \frac{5A + A^2 B - A^2 - B^2}{B(1-A)} \\ &= \frac{4A}{1-A} = \frac{1-X^2}{X} \cdot \frac{1-4X-X^2}{1-X^2} = \frac{1}{X} - 4 - X = \frac{f_1^3 f_5}{q f_2 f_{10}^3} \end{aligned}$$

as desired. □

## Lemma 9

We have the identity

$$\left( \frac{f_4 f_{10}^{12}}{f_1^2 f_5^2 f_{20}^5} - 4q^3 \frac{f_2^3 f_5^3 f_{20}^5}{f_1^3 f_4 f_{10}^3} \right)^2 = \frac{f_2^4 f_5^{12}}{f_1^4 f_{10}^4} + 4q^2 \frac{f_2^2 f_5^2 f_{10}^6}{f_1^2}.$$

*Proof.* We know from Lemma 7 that

$$\begin{aligned} \left( \frac{f_4 f_{10}^{12}}{f_1^2 f_5^2 f_{20}^5} - 4q^3 \frac{f_2^3 f_5^3 f_{20}^5}{f_1^3 f_4 f_{10}^3} \right)^2 &= \left( \frac{f_4 f_{10}^{12}}{f_1^2 f_5^2 f_{20}^5} + 4q^3 \frac{f_2^3 f_5^3 f_{20}^5}{f_1^3 f_4 f_{10}^3} \right)^2 - 16q^3 \frac{f_2^3 f_5 f_{10}^9}{f_1^5} \\ &= \left( \frac{f_2^3 f_5^8}{f_1^4 f_{10}^3} - 2q \frac{f_2^2 f_5^3 f_{10}^2}{f_1^3} \right)^2 - 16q^3 \frac{f_2^3 f_5 f_{10}^9}{f_1^5}. \end{aligned} \quad (26)$$

*Proof (continued).* By expanding and applying (9) and (11), we find that

$$\begin{aligned}
 & \frac{f_1^5}{q^3 f_2^3 f_5 f_{10}^9} \left( \frac{f_2^3 f_5^8}{f_1^4 f_{10}^3} - 2q \frac{f_2^2 f_5^3 f_{10}^2}{f_1^3} \right)^2 - 16 \\
 &= \left( \frac{f_2 f_5^5}{q f_1 f_{10}^5} \right)^3 - 4 \left( \frac{f_2 f_5^5}{q f_1 f_{10}^5} \right)^2 + 4 \frac{f_2 f_5^5}{q f_1 f_{10}^5} - 16 = \left[ \left( \frac{f_2 f_5^5}{q f_1 f_{10}^5} \right)^2 + 4 \right] \left( \frac{f_2 f_5^5}{q f_1 f_{10}^5} - 4 \right) \\
 &= \left( \frac{f_2^4 f_5^{10}}{q^2 f_1^2 f_{10}^{10}} + 4 \right) \frac{f_1^3 f_5}{q f_2 f_{10}^3} = \frac{f_1 f_2 f_5^{11}}{q^3 f_{10}^{13}} + 4 \frac{f_1^3 f_5}{q f_2 f_{10}^3}. \tag{27}
 \end{aligned}$$

Multiplying both sides of (27) by  $q^3 f_2^3 f_5 f_{10}^9 / f_1^5$  and comparing with (26), we obtain the desired identity.  $\square$

## Lemma 10

We have the identity

$$-q \left( \frac{f_5^5}{f_1} + 2q \frac{f_{10}^5}{f_2} \right)^2 + f_1^4 f_5^4 + 9q \frac{f_5^{10}}{f_1^2} - 8q^3 \frac{f_{10}^{10}}{f_2^2} = \frac{f_2^4 f_5^{12}}{f_1^4 f_{10}^4} + 4q^2 \frac{f_2^2 f_5^2 f_{10}^6}{f_1^2}.$$

*Proof.* Expanding the left-hand side of the given identity and then dividing both sides by  $q^3 f_2^3 f_5 f_{10}^9 / f_1^5$ , it remains to prove that

$$\frac{f_1^9 f_5^3}{q^3 f_2^3 f_{10}^9} + 8 \frac{f_1^3 f_5^9}{q^2 f_2^3 f_{10}^9} - 4 \frac{f_1^4 f_5^4}{q f_2^4 f_{10}^4} - 12 \frac{f_1^5 f_{10}}{f_2^5 f_5} = \frac{f_1 f_2 f_5^{11}}{q^3 f_{10}^{13}} + 4 \frac{f_1^3 f_5}{q f_2 f_{10}^3}. \quad (28)$$

*Proof (continued).* We simplify the left-hand side of (28) as follows. Letting  $X := k(q)$ , we infer from (9)–(11) that

$$\frac{f_1^9 f_5^3}{q^3 f_2^3 f_{10}^9} = \left( \frac{f_1^3 f_5}{q f_2 f_{10}^3} \right)^3 = \left( \frac{1}{X} - 4 - X \right)^3, \quad (29)$$

$$\frac{f_1^3 f_5^9}{q^2 f_2^3 f_{10}^9} = \frac{f_1^3 f_5}{q f_2 f_{10}^3} \cdot \frac{q f_1^2 f_{10}^4}{f_2^4 f_5^2} \left( \frac{f_2 f_5^5}{q f_1 f_{10}^5} \right)^2 = \left( \frac{1}{X} - 4 - X \right) \frac{X}{1 + X - X^2} \left( \frac{1}{X} - X \right)^2, \quad (30)$$

$$\frac{f_1^4 f_5^4}{q f_2^4 f_{10}^4} = \frac{f_1^3 f_5}{q f_2 f_{10}^3} \cdot \frac{q f_1^2 f_{10}^4}{f_2^4 f_5^2} \cdot \frac{f_2 f_5^5}{q f_1 f_{10}^5} = \left( \frac{1}{X} - 4 - X \right) \frac{X}{1 + X - X^2} \left( \frac{1}{X} - X \right), \quad (31)$$

$$\frac{f_1^5 f_{10}}{f_2^5 f_5} = \frac{f_1^3 f_5}{q f_2 f_{10}^3} \cdot \frac{q f_1^2 f_{10}^4}{f_2^4 f_5^2} = \left( \frac{1}{X} - 4 - X \right) \frac{X}{1 + X - X^2}. \quad (32)$$

Applying (29)–(32) and (9)–(11), we find that

$$\begin{aligned}
 & \frac{f_1^9 f_5^3}{q^3 f_2^3 f_{10}^9} + 8 \frac{f_1^3 f_5^9}{q^2 f_2^3 f_{10}^9} - 4 \frac{f_1^4 f_5^4}{q f_2^4 f_{10}^4} - 12 \frac{f_1^5 f_{10}}{f_2^5 f_5} \\
 &= \left(\frac{1}{X} - 4 - X\right)^3 + \left(\frac{1}{X} - 4 - X\right) \frac{4X}{1 + X - X^2} \left[2 \left(\frac{1}{X} - X\right)^2 - \left(\frac{1}{X} - X\right) - 3\right] \\
 &= \left(\frac{1}{X} - 4 - X\right)^3 + \left(\frac{1}{X} - 4 - X\right) \frac{4X}{1 + X - X^2} \left(\frac{2}{X} - 3 - 2X\right) \left(\frac{1}{X} + 1 - X\right) \\
 &= \left(\frac{1}{X} - 4 - X\right) \left[\left(\frac{1}{X} - 4 - X\right)^2 + \frac{8}{X} - 12 - 8X\right] \\
 &= \left(\frac{1}{X} - 4 - X\right) \left[\left(\frac{1}{X} - X\right)^2 + 4\right] = \frac{f_1^3 f_5}{q f_2 f_{10}^3} \left(\frac{f_2^4 f_5^{10}}{q^2 f_1^2 f_{10}^{10}} + 4\right) = \frac{f_1 f_2 f_5^{11}}{q^3 f_{10}^{13}} + 4 \frac{f_1^3 f_5}{q f_2 f_{10}^3},
 \end{aligned}$$

which is exactly the right-hand side of (28). □

As an application of Lemmas 6–10, we provide two generating function formulas needed to prove Theorems 1 and 2.

### Theorem 11

*We have that*

$$\sum_{n=0}^{\infty} A_5(2n)q^n = f_1^4 f_5^4 + 9q \frac{f_5^{10}}{f_1^2} - 8q^3 \frac{f_{10}^{10}}{f_2^2}.$$

Using (7) and (8), we expand

$$\sum_{n=0}^{\infty} A_5(n)q^n = \frac{f_5^{10}}{f_1^2} = \frac{f_5^2}{f_1^2} \cdot f_5^8 = \left( \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q \frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}} \right)^2 \left( \frac{f_{20}^{10}}{f_{10}^2 f_{40}^4} - 4q^5 \frac{f_{10}^2 f_{40}^4}{f_{20}^2} \right)^2. \quad (33)$$

We consider the terms in the expansion of (31) containing  $q^{2n}$ . Applying Lemmas 6, 9, and 10, we find that

$$\begin{aligned} \sum_{n=0}^{\infty} A_5(2n)q^n &= \frac{f_4^2 f_{10}^{24}}{f_1^4 f_5^4 f_{20}^{10}} + q \frac{f_2^6 f_{10}^{18}}{f_1^6 f_4^2 f_5^2 f_{20}^6} - 16q^3 \frac{f_2^3 f_5 f_{10}^9}{f_1^5} + 16q^5 \frac{f_4^2 f_5^4 f_{20}^6}{f_1^4} + 16q^6 \frac{f_2^6 f_5^6 f_{20}^{10}}{f_1^6 f_4^2 f_{10}^6} \\ &= \left( \frac{f_4 f_{10}^{12}}{f_1^2 f_5^2 f_{20}^5} - 4q^3 \frac{f_2^3 f_5^3 f_{20}^5}{f_1^3 f_4 f_{10}^3} \right)^2 + q \left( \frac{f_2^3 f_{10}^9}{f_1^3 f_4 f_5 f_{20}^3} - 4q^2 \frac{f_4 f_5^2 f_{20}^3}{f_1^2} \right)^2 \\ &= \frac{f_2^4 f_5^{12}}{f_1^4 f_{10}^4} + 4q^2 \frac{f_2^2 f_5^2 f_{10}^6}{f_1^2} + q \left( \frac{f_5^5}{f_1} + 2q \frac{f_{10}^5}{f_2} \right)^2 \\ &= f_1^4 f_5^4 + 9q \frac{f_5^{10}}{f_1^2} - 8q^3 \frac{f_{10}^{10}}{f_2^2}. \end{aligned}$$

### Theorem 12

Let  $q^{-3}f_1^4 f_5^4 := \sum_{n=-3}^{\infty} c(n+3)q^n$ . Then

$$\sum_{n=-1}^{\infty} c(2n+3)q^n = -4 \frac{f_1^4 f_5^4}{q} - 8f_2^4 f_{10}^4.$$

*Proof.* Using (7), we write

$$\sum_{n=-3}^{\infty} c(n+3)q^n = \frac{f_1^4 f_5^4}{q^3} = \frac{1}{q^3} \left( \frac{f_4^{10}}{f_2^2 f_8^4} - 4q \frac{f_2^2 f_8^4}{f_4^2} \right) \left( \frac{f_{20}^{10}}{f_{10}^2 f_{40}^4} - 4q^5 \frac{f_{10}^2 f_{40}^4}{f_{20}^2} \right). \quad (34)$$

We extract the terms in the expansion of (34) involving  $q^{2n}$ , so that

$$\sum_{n=-1}^{\infty} c(2n+3)q^n = -4 \frac{f_1^2 f_4^4 f_{10}^{10}}{q f_2^2 f_5^2 f_{20}^4} - 4q \frac{f_2^{10} f_5^2 f_{20}^4}{f_1^2 f_4^4 f_{10}^2}. \quad (35)$$

*Proof (continued).* We now multiply both sides of Lemma 8 by  $f_2 f_5 f_{10}^3 / f_1$  and then square both sides of the resulting expression. We obtain

$$\frac{f_1^4 f_5^4}{q^2} = \left( \frac{f_1 f_4^2 f_{10}^5}{q f_2 f_5 f_{20}^2} - \frac{f_2^5 f_5 f_{20}^2}{f_1 f_4^2 f_{10}} \right)^2 = \frac{f_1^2 f_4^4 f_{10}^{10}}{q^2 f_2^2 f_5^2 f_{20}^4} - 2 \frac{f_2^4 f_{10}^4}{q} + \frac{f_2^{10} f_5^2 f_{20}^4}{f_1^2 f_4^4 f_{10}^2}. \quad (36)$$

We infer from (35) and (36) that

$$\begin{aligned} \sum_{n=-1}^{\infty} c(2n+3)q^n &= -4q \left( \frac{f_1^2 f_4^4 f_{10}^{10}}{q^2 f_2^2 f_5^2 f_{20}^4} + \frac{f_2^{10} f_5^2 f_{20}^4}{f_1^2 f_4^4 f_{10}^2} \right) \\ &= -4q \left( \frac{f_1^4 f_5^4}{q^2} + 2 \frac{f_2^4 f_{10}^4}{q} \right) \\ &= -4 \frac{f_1^4 f_5^4}{q} - 8 f_2^4 f_{10}^4. \end{aligned}$$

*Proof of Theorem 1.* We proceed by induction on  $k$ . From Theorem 11, we have that

$$\sum_{n=-1}^{\infty} A_5(2n+2)q^n = \frac{f_1^4 f_5^4}{q} + 9 \frac{f_5^{10}}{f_1^2} - 8q^2 \frac{f_{10}^{10}}{f_2^2}, \quad (37)$$

so (1) holds for  $k = 1$ . Suppose now that (1) holds for some  $k \geq 1$ . We divide both sides of (1) by  $q^2$  so that

$$\begin{aligned} & \sum_{n=-3}^{\infty} A_5(2^k n + 2^{k+2} - 2)q^n \\ &= B_k \frac{f_1^4 f_5^4}{q^3} - 8B_{k-1} \frac{f_2^4 f_{10}^4}{q^2} + \frac{8^{k+1} - 1}{7} \cdot \frac{f_5^{10}}{q^2 f_1^2} - \frac{8^{k+1} - 8}{7} \cdot \frac{f_{10}^{10}}{f_2^2} \\ &= B_k \sum_{n=-3}^{\infty} c(n+3)q^n - 8B_{k-1} \frac{f_2^4 f_{10}^4}{q^2} + \frac{8^{k+1} - 1}{7q^2} \sum_{n=0}^{\infty} A_5(n)q^n - \frac{8^{k+1} - 8}{7} \cdot \frac{f_{10}^{10}}{f_2^2}. \end{aligned} \quad (38)$$

*Proof (continued).* We extract the terms in the expansion of (38) involving  $q^{2n}$ . We deduce from Theorems 11 and 12 that

$$\begin{aligned}
 & \sum_{n=-1}^{\infty} A_5(2^{k+1}n + 2^{k+2} - 2)q^n \\
 &= B_k \sum_{n=-1}^{\infty} c(2n+3)q^n - 8B_{k-1} \frac{f_1^4 f_5^4}{q} + \frac{8^{k+1} - 1}{7q} \sum_{n=0}^{\infty} A_5(2n)q^n - \frac{8^{k+1} - 8}{7} \cdot \frac{f_5^{10}}{f_1^2} \\
 &= B_k \left( -4 \frac{f_1^4 f_5^4}{q} - 8f_2^4 f_{10}^4 \right) - 8B_{k-1} \frac{f_1^4 f_5^4}{q} + \frac{8^{k+1} - 1}{7q} \left( f_1^4 f_5^4 + 9q \frac{f_5^{10}}{f_1^2} - 8q^3 \frac{f_{10}^{10}}{f_2^2} \right) \\
 &\quad - \frac{8^{k+1} - 8}{7} \cdot \frac{f_5^{10}}{f_1^2} \\
 &= B_{k+1} \frac{f_1^4 f_5^4}{q} - 8B_k f_2^4 f_{10}^4 + \frac{8^{k+2} - 1}{7} \cdot \frac{f_5^{10}}{f_1^2} - \frac{8^{k+2} - 8}{7} \cdot \frac{q^2 f_{10}^{10}}{f_2^2}.
 \end{aligned}$$

Thus, (1) holds for  $k+1$ , so it holds for all  $k \geq 1$  by induction.  $\square$

To prove Theorem 2, we need to show the following result on certain values of  $B_k$ .

### Lemma 13

For all  $m \geq 0$ , we have  $B_{4m+3} = \frac{8^{4m+4} - 1}{91}$ .

*Proof.* Observe that for  $m \geq 0$ ,

$$\begin{aligned}
 B_{4m+7} + 64B_{4m+3} &= B_{4m+7} + 4B_{4m+6} + 8B_{4m+5} - 4(B_{4m+6} + 4B_{4m+5} + 8B_{4m+4}) \\
 &\quad + 8(B_{4m+5} + 4B_{4m+4} + 8B_{4m+3}) \\
 &= \frac{8^{4m+7} - 1}{7} - 4 \cdot \frac{8^{4m+6} - 1}{7} + 8 \cdot \frac{8^{4m+5} - 1}{7} = \frac{5(8^{4m+6} - 1)}{7}.
 \end{aligned} \tag{39}$$

By the theory of linear recurrences, we obtain

$$B_{4m+3} = A \cdot 8^{4m} + B(-64)^m + C$$

for all  $m \geq 0$  and some constants  $A$ ,  $B$ , and  $C$ . Using the recurrence relation for  $B_k$ , we have  $B_3 = 45$ , and employing (39), we compute  $B_7 = 184365$  and  $B_{11} = 755159085$ . Thus, we find that  $(A, B, C) = (4096/91, 0, -1/91)$ , leading us to the desired value of  $B_{4m+3}$ .  $\square$

*Proof of Theorem 2.* We divide both sides of (1) by  $q$  so that

$$\sum_{n=-2}^{\infty} A_5(2^k n + 3 \cdot 2^k - 2)q^n = B_k \frac{f_1^4 f_5^4}{q^2} - 8B_{k-1} \frac{f_2^4 f_{10}^4}{q} + \frac{8^{k+1} - 1}{7} \cdot \frac{f_5^{10}}{qf_1^2} - \frac{8^{k+1} - 8}{7} \cdot \frac{qf_{10}^{10}}{f_2^2}. \quad (40)$$

Define  $q^{-2}f_1^4 f_5^4 := \sum_{n=-2}^{\infty} c(n+2)q^n$ . Observe that the  $q$ -expansions of  $q^{-1}f_2^4 f_{10}^4$  and  $qf_{10}^{10}/f_2^2$  contain only terms with odd exponents. Thus, by looking at the terms in the expansion of (40) involving  $q^{2n}$ , we see that

$$\sum_{n=-1}^{\infty} A_5(2^{k+1}n + 3 \cdot 2^k - 2)q^n = B_k \sum_{n=-1}^{\infty} c(2n+2)q^n + \frac{8^{k+1} - 1}{7} \sum_{n=0}^{\infty} A_5(2n+1)q^n. \quad (41)$$

We next divide both sides of (37) by  $q$  and consider terms in the resulting expansion involving  $q^{2n}$ . We get

$$\sum_{n=-1}^{\infty} A_5(4n+4)q^n = \sum_{n=-1}^{\infty} c(2n+2)q^n + 9 \sum_{n=0}^{\infty} A_5(2n+1)q^n. \quad (42)$$

*Proof (continued).* Multiplying both sides of (42) by  $B_k$  and subtracting from (41), we obtain

$$\sum_{n=-1}^{\infty} (A_5(2^{k+1}n + 3 \cdot 2^k - 2) - B_k A_5(4n + 4))q^n = \left( \frac{8^{k+1} - 1}{7} - 9B_k \right) \sum_{n=0}^{\infty} A_5(2n + 1)q^n.$$

Comparing the coefficients of  $q^n$  for  $n \geq 0$  on both sides of the above expression yields (2).

We now set  $k = 4m + 3$  in (2) and use Lemma 13. Since

$$\frac{8^{4m+4} - 1}{7} - 9B_{4m+3} = \frac{8^{4m+4} - 1}{7} - \frac{9(8^{4m+4} - 1)}{91} = \frac{4(8^{4m+4} - 1)}{91},$$

we finally arrive at

$$A_5(2^{4m+4}n + 3 \cdot 2^{4m+3} - 2) = \frac{8^{4m+4} - 1}{91} (A_5(4n + 4) + 4A_5(2n + 1))$$

for all  $m \geq 0$  and  $n \geq 0$ , which immediately proves (3). □

We now require the following 3-dissections

$$\frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q \frac{f_{18}^2}{f_9}, \quad (43)$$

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q \frac{f_6^3 f_9^3}{f_3^7} + 4q^2 \frac{f_6^2 f_{18}^3}{f_3^6}. \quad (44)$$

Identity (43) is labeled as (22.6.13) on page 195 of Hirschhorn's book (2017). The proof of (44) can be found on page 68 of Hirschhorn and Sellers (2005).

We next provide the following identities involving  $C(q)$ :

$$\frac{f_2^8 f_3^4}{q f_1^4 f_6^8} = \frac{1}{q C^3(q)} + 1, \quad (45)$$

$$\frac{f_1^4 f_2^4}{q f_3^4 f_6^4} = \frac{1}{q C^3(q)} - 7 - 8q C^3(q), \quad (46)$$

$$f_1 f_2 = f_9 f_{18} \left( \frac{1}{C(q^3)} - q - 2q^2 C(q^3) \right), \quad (47)$$

$$\frac{1}{f_1 f_2} = \frac{f_9^3 f_{18}^3}{f_3^4 f_6^4} \left( \frac{1}{C^2(q^3)} + \frac{q}{C(q^3)} + 3q^2 - 2q^3 C(q^3) + 4q^2 C^2(q^3) \right). \quad (48)$$

Identities (45) and (46) can be found in Theorems 6.9 and 6.11 of Cooper's book (2017), respectively. Identity (47) follows from replacing  $q$  with  $q^3$  in Theorem 2 of Chan (2010). For the proof of (48), see the proof of Theorem 1 of Chan (2010).

We also need the following identities involving  $h(q)$ , which can be found in Theorem 12.2 of Cooper's book (2017):

$$\frac{f_3^3 f_4}{q f_1 f_{12}^3} = \frac{1}{h(q)} + h(q), \quad (49)$$

$$\frac{f_4^4 f_6^2}{q f_2^2 f_{12}^4} = \frac{1}{h(q)} - 1 + h(q), \quad (50)$$

$$\frac{f_1 f_4^2 f_6^9}{q f_2^3 f_3^3 f_{12}^6} = \frac{1}{h(q)} - 2 + h(q), \quad (51)$$

$$\frac{f_2^7 f_3}{q f_1^3 f_4^2 f_6 f_{12}^2} = \frac{1}{h(q)} + 2 + h(q). \quad (52)$$

## Lemma 14

We have the identities

$$\frac{f_2^7 f_3^5}{f_1^5 f_6} = \frac{f_2^2 f_3^{10}}{f_1^4 f_6^2} + q \frac{f_3 f_6^7}{f_1 f_2}, \quad (53)$$

$$\frac{f_1^3 f_2^3 f_6^3}{f_3^3} = -8 \frac{f_2^{12}}{f_1^6} + 9 \left( \frac{f_2^2 f_3^{10}}{f_1^4 f_6^2} + q \frac{f_3 f_6^7}{f_1 f_2} \right). \quad (54)$$

*Proof.* Multiplying both sides of (45) by  $qf_3f_6^7/(f_1f_2)$  yields (53). On the other hand, dividing both sides of (54) by  $qf_3f_6^7/(f_1f_2)$ , it suffices to show that

$$\frac{f_1^4f_2^4}{qf_3^4f_6^4} + 8\frac{f_2^{13}}{qf_1^5f_3f_6^7} = 9\frac{f_2^3f_3^9}{qf_1^3f_6^9} + 9. \quad (55)$$

Observe that from (45), we have

$$\frac{f_2^{13}}{qf_1^5f_3f_6^7} = \left(\frac{f_2^8f_3^4}{qf_1^4f_6^8}\right)^2 \cdot \frac{qf_1^3f_6^9}{f_2^3f_3^9} = \left(\frac{1}{qC^3(q)} + 1\right)^2 \cdot qC^3(q). \quad (56)$$

Combining (46) and (56) leads us to

$$\begin{aligned} \frac{f_1^4f_2^4}{qf_3^4f_6^4} + 8\frac{f_2^{13}}{qf_1^5f_3f_6^7} &= \frac{1}{qC^3(q)} - 7 - 8qC^3(q) + 8\left(\frac{1}{qC^3(q)} + 2 + qC^3(q)\right) \\ &= \frac{9}{qC^3(q)} + 9 = 9\frac{f_2^3f_3^9}{qf_1^3f_6^9} + 9, \end{aligned}$$

which is precisely (55). □

### Lemma 15

We have the identity

$$\frac{f_2^{15} f_3^2 f_{12}}{q f_1^6 f_4^3 f_6^9} + 4q^2 \frac{f_2^3 f_{12}^9}{f_4^3 f_6^9} = \frac{f_3^3 f_4}{q f_1 f_{12}^3} + 2 + 3 \frac{f_2^3 f_3^6 f_{12}^3}{f_1^2 f_4 f_6^9} + 4q \frac{f_2^3 f_3^3 f_{12}^6}{f_1 f_4^2 f_6^9}.$$

*Proof.* We rewrite the given identity as

$$\frac{f_2^{15} f_3^2 f_{12}}{q f_1^6 f_4^3 f_6^9} - 3 \frac{f_2^3 f_3^6 f_{12}^3}{f_1^2 f_4 f_6^9} - 4q \frac{f_2^3 f_3^3 f_{12}^6}{f_1 f_4^2 f_6^9} + 4q^2 \frac{f_2^3 f_{12}^9}{f_4^3 f_6^9} = \frac{f_3^3 f_4}{q f_1 f_{12}^3} + 2. \quad (57)$$

*Proof (continued).* Letting  $h := h(q)$ , we write each summand on the left-hand side of (57) using (49)–(52). We find that

$$\begin{aligned} \frac{f_2^{15} f_3^2 f_{12}}{q f_1^6 f_4^3 f_6^9} &= \frac{f_4^4 f_6^2}{q f_2^2 f_{12}^4} \cdot \frac{q f_1 f_{12}^3}{f_3^3 f_4} \left( \frac{f_2^7 f_3}{q f_1^3 f_4^2 f_6 f_{12}^2} \right)^2 \frac{q f_2^3 f_3^3 f_{12}^6}{f_1 f_4^2 f_6^9} \\ &= \left( \frac{1}{h} - 1 + h \right) \left( \frac{1}{h} + h \right)^{-1} \left( \frac{1}{h} + 2 + h \right)^2 \left( \frac{1}{h} - 2 + h \right)^{-1} \\ &= \frac{(1+h)^4 (1-h+h^2)}{h(1-h)^2 (1+h^2)}, \end{aligned} \tag{58}$$

$$\begin{aligned} \frac{f_2^3 f_3^6 f_{12}^3}{f_1^2 f_4 f_6^9} &= \frac{f_3^3 f_4}{q f_1 f_{12}^3} \cdot \frac{q f_2^3 f_3^3 f_{12}^6}{f_1 f_4^2 f_6^9} = \left( \frac{1}{h} + h \right) \left( \frac{1}{h} - 2 + h \right)^{-1} \\ &= \frac{1+h^2}{(1-h)^2}, \end{aligned} \tag{59}$$

*Proof (continued).*

$$q \frac{f_2^3 f_3^3 f_{12}^6}{f_1 f_4^2 f_6^9} = \left( \frac{1}{h} - 2 + h \right)^{-1} = \frac{h}{(1-h)^2}, \quad (60)$$

$$\begin{aligned} q^2 \frac{f_2^3 f_{12}^9}{f_4^3 f_6^9} &= \frac{q f_1 f_{12}^3}{f_3^3 f_4} \cdot \frac{q f_2^3 f_3^3 f_{12}^6}{f_1 f_4^2 f_6^9} = \left( \frac{1}{h} + h \right)^{-1} \left( \frac{1}{h} - 2 + h \right)^{-1} \\ &= \frac{h^2}{(1-h)^2(1+h^2)}. \end{aligned} \quad (61)$$

*Proof (continued).* We infer from (49) and (58)–(61) that

$$\begin{aligned}
 & \frac{f_2^{15} f_3^2 f_{12}}{q f_1^6 f_4^3 f_6^9} - 3 \frac{f_2^3 f_3^6 f_{12}^3}{f_1^2 f_4 f_6^9} - 4q \frac{f_2^3 f_3^3 f_{12}^6}{f_1 f_4^2 f_6^9} + 4q^2 \frac{f_2^3 f_{12}^9}{f_4^3 f_6^9} \\
 &= \frac{(1+h)^4}{h(1-h)^2} - \frac{(1+h)^4}{(1-h)^2(1+h^2)} - \frac{3(1+h^2)}{(1-h)^2} - \frac{4h}{(1-h)^2} + \frac{4h^2}{(1-h)^2(1+h^2)} \\
 &= \frac{(1+h)^4}{h(1-h)^2} - \frac{3h^2+4h+3}{(1-h)^2} + \frac{(2h+(1+h)^2)(2h-(1+h)^2)}{(1-h)^2(1+h^2)} \\
 &= \frac{(1+h)^4}{h(1-h)^2} - \frac{3h^2+4h+3}{(1-h)^2} - \frac{h^2+4h+1}{(1-h)^2} \\
 &= \frac{(1+h)^4}{h(1-h)^2} - \frac{4(1+h)^2}{(1-h)^2} = \left(\frac{1+h}{1-h}\right)^2 \left(\frac{(1+h)^2}{h} - 4\right) \\
 &= \left(\frac{1+h}{1-h}\right)^2 \left(\frac{(1-h)^2}{h}\right) = \frac{1}{h} + h + 2 = \frac{f_3^3 f_4}{q f_1 f_{12}^3} + 2,
 \end{aligned}$$

which is the right-hand side of (57). □

We now use the previous identities and Lemmas 14 and 15 to deduce the generating functions for  $A_4(3n + 1)$  and  $A_4(27n + 19)$ , which will be instrumental to the proofs of Theorem 3 and 4.

### Theorem 16

*We have that*

$$\sum_{n=0}^{\infty} A_4(3n + 1)q^n = 2 \frac{f_2^{12}}{f_1^6} + 9q^3 \frac{f_{12}^8}{f_3^2}.$$

*Proof.* Using (43) and (44), we write

$$\sum_{n=0}^{\infty} A_4(n)q^n = \frac{f_4^8}{f_1^2} = \left(\frac{f_4^2}{f_2}\right)^4 \left(\frac{f_2^2}{f_1}\right)^2 = \left(\frac{f_{12}f_{18}^2}{f_6f_{36}} + q^2\frac{f_{36}^2}{f_{18}}\right)^4 \left(\frac{f_6f_9^2}{f_3f_{18}} + q\frac{f_{18}^2}{f_9}\right)^2. \quad (62)$$

We consider the terms involving  $q^{3n+1}$  in the expansion of (62). In view of Lemma 15, we arrive at

$$\begin{aligned} \sum_{n=0}^{\infty} A_4(3n+1)q^n &= 2\frac{f_3f_4^4f_6^9}{f_1f_2^3f_{12}^4} + 4q\frac{f_4^3f_6^9}{f_2^3f_3^2f_{12}} + 6q\frac{f_3^4f_4^2f_{12}^2}{f_1^2} + 8q^2\frac{f_3f_4f_{12}^5}{f_1} + q^3\frac{f_{12}^8}{f_3^2} \\ &= 2q\frac{f_4^3f_6^9}{f_2^3f_3^2f_{12}} \left( \frac{f_3^3f_4}{qf_1f_{12}^3} + 2 + 3\frac{f_2^3f_3^6f_{12}^3}{f_1^2f_4f_6^9} + 4q\frac{f_2^3f_3^3f_{12}^6}{f_1f_4^2f_6^9} \right) + q^3\frac{f_{12}^8}{f_3^2} \\ &= 2q\frac{f_4^3f_6^9}{f_2^3f_3^2f_{12}} \left( \frac{f_2^{15}f_3^2f_{12}}{qf_1^6f_4^3f_6^9} + 4q^2\frac{f_2^3f_{12}^9}{f_4^3f_6^9} \right) + q^3\frac{f_{12}^8}{f_3^2} \\ &= 2\frac{f_2^{12}}{f_1^6} + 8q^3\frac{f_{12}^8}{f_3^2} + q^3\frac{f_{12}^8}{f_3^2} = 2\frac{f_2^{12}}{f_1^6} + 9q^3\frac{f_{12}^8}{f_3^2}. \end{aligned}$$

## Theorem 17

For all integers  $n \geq 0$  with  $3 \nmid n$ , we have  $A_4(3n + 1) \equiv 0 \pmod{6}$ .

*Proof.* Using (43), we express Theorem 16 as

$$\sum_{n=0}^{\infty} A_4(3n + 1)q^n = 2 \left( \frac{f_2^2}{f_1} \right)^6 + 9q^3 \frac{f_{12}^8}{f_3^2} = 2 \left( \frac{f_6 f_9^2}{f_3 f_{18}} + q \frac{f_{18}^2}{f_9} \right)^6 + 9q^3 \frac{f_{12}^8}{f_3^2}. \quad (63)$$

We extract the terms involving  $q^{3n+1}$  and  $q^{3n+2}$  in the expansion of (63). We deduce that

$$\begin{aligned} \sum_{n=0}^{\infty} A_4(9n + 4)q^n &= 6 \left( 2 \frac{f_2^5 f_3^9}{f_1^5 f_6^3} + 5q \frac{f_2^2 f_6^6}{f_1^2} \right), \\ \sum_{n=0}^{\infty} A_4(9n + 7)q^n &= 6 \left( 5 \frac{f_2^4 f_3^6}{f_1^4} + 2q \frac{f_2 f_6^9}{f_1 f_3^3} \right), \end{aligned}$$

which immediately yield the desired congruence. □

Given a  $q$ -series  $f(q) := \sum_{n=n_0}^{\infty} a(n)q^n \in \mathbb{Z}[[q]]$ , we define the linear maps in  $\mathbb{Z}[[q]]$  by

$$U(f(q)) := \sum_{n=\lceil n_0/3 \rceil}^{\infty} a(3n)q^n,$$

$$V(f(q)) := \sum_{n=\lceil (n_0-1)/3 \rceil}^{\infty} a(3n+1)q^n.$$

The next two lemmas provide the closed-form formulas for the images of certain  $q$ -series under  $U$  and  $V$ . This will play a key role in determining the generating function for  $A_4(27n+19)$ .

## Lemma 18

We have the identities

$$U \left( \frac{f_2^{12}}{q^3 f_1^6} \right) = \frac{f_2^6 f_3^{12}}{q f_1^6 f_6^6} + 20 \frac{f_2^3 f_3^3 f_6^3}{f_1^3} + q \frac{f_6^{12}}{f_3^6}, \quad (64)$$

$$U \left( \frac{f_2^2 f_3^{10}}{q^3 f_1^4 f_6^2} \right) = \frac{f_2^6 f_3^{12}}{q f_1^6 f_6^6} + 16 \frac{f_2^3 f_3^3 f_6^3}{f_1^3}, \quad (65)$$

$$U \left( \frac{f_3 f_6^7}{q^2 f_1 f_2} \right) = 3 \frac{f_2^3 f_3^3 f_6^3}{f_1^3}. \quad (66)$$

*Proof.* We use (43) and (44) to write

$$\frac{f_2^{12}}{q^3 f_1^6} = \frac{1}{q^3} \left( \frac{f_2^2}{f_1} \right)^6 = \frac{1}{q^3} \left( \frac{f_6 f_9^2}{f_3 f_{18}} + q \frac{f_{18}^2}{f_9} \right)^6. \quad (67)$$

Examining the terms involving  $q^{3n}$  in the expansion of (67), we get (64). We next write

$$\frac{f_2^2 f_3^{10}}{q^3 f_1^4 f_6^2} = \frac{f_3^{10}}{q^3 f_6^2} \left( \frac{f_2}{f_1^2} \right)^2 = \frac{f_3^{10}}{q^3 f_6^2} \left( \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q \frac{f_6^3 f_9^3}{f_3^7} + 4q^2 \frac{f_6^2 f_{18}^3}{f_3^6} \right)^2. \quad (68)$$

Getting the terms involving  $q^{3n}$  in the expansion of (68), we obtain (65). We finally use (48) so that

$$\frac{f_3 f_6^7}{q^2 f_1 f_2} = \frac{f_3 f_6^7}{q^2} \cdot \frac{f_9^3 f_{18}^3}{f_3^4 f_6^4} \left( \frac{1}{C^2(q^3)} + \frac{q}{C(q^3)} + 3q^2 - 2q^3 C(q^3) + 4q^2 C^2(q^3) \right). \quad (69)$$

Examining the terms involving  $q^{3n}$  in the expansion of (69), we arrive at (66).  $\square$

## Lemma 19

We have the identities

$$V \left( \frac{f_2^6 f_3^{12}}{q f_1^6 f_6^6} \right) = 21 \frac{f_2^2 f_3^{10}}{f_1^4 f_6^2} + 48q \frac{f_3 f_6^7}{f_1 f_2}, \quad (70)$$

$$V \left( \frac{f_2^3 f_3^3 f_6^3}{f_1^3} \right) = 3 \frac{f_2^2 f_3^{10}}{f_1^4 f_6^2} + 3q \frac{f_3 f_6^7}{f_1 f_2}. \quad (71)$$

*Proof.* We know from (43) and (44) that

$$\frac{f_2^6 f_3^{12}}{q f_1^6 f_6^6} = \frac{f_3^{12}}{q f_6^6} \left( \frac{f_2^2}{f_1} \right)^2 \left( \frac{f_2}{f_1^2} \right)^2 = \frac{f_3^{12}}{q f_6^6} \left( \frac{f_6 f_9^2}{f_3 f_{18}} + q \frac{f_{18}^2}{f_9} \right)^2 \left( \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q \frac{f_6^3 f_9^3}{f_3^7} + 4q^2 \frac{f_6^2 f_{18}^3}{f_3^6} \right)^2. \quad (72)$$

Considering the terms involving  $q^{3n+1}$  in the expansion of (72) yields (70). We also know that

$$\frac{f_2^3 f_3^3 f_6^3}{f_1^3} = f_3^3 f_6^3 \left( \frac{f_2^2}{f_1} \right) \left( \frac{f_2}{f_1^2} \right) = f_3^3 f_6^3 \left( \frac{f_6 f_9^2}{f_3 f_{18}} + q \frac{f_{18}^2}{f_9} \right) \left( \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q \frac{f_6^3 f_9^3}{f_3^7} + 4q^2 \frac{f_6^2 f_{18}^3}{f_3^6} \right). \quad (73)$$

Considering the terms involving  $q^{3n+1}$  in the expansion of (73), we see that

$$V \left( \frac{f_2^3 f_3^3 f_6^3}{f_1^3} \right) = 3 \frac{f_2^7 f_3^5}{f_1^5 f_6}.$$

Thus, (71) follows from the above expression and (53).

## Theorem 20

We have that

$$\sum_{n=0}^{\infty} A_4(27n + 19)q^n = 20 \frac{f_2^{12}}{f_1^6} + 54 \left( 3 \frac{f_2^2 f_3^{10}}{f_1^4 f_6^2} + 4q \frac{f_3 f_6^7}{f_1 f_2} \right) + 81q^3 \frac{f_{12}^8}{f_3^2}.$$

*Proof.* Dividing both sides of Theorem 16 by  $q^3$  yields

$$\sum_{n=-3}^{\infty} A_4(3n+10)q^n = 2\frac{f_2^{12}}{q^3 f_1^6} + 9\frac{f_3^8}{f_3^2}. \quad (74)$$

Applying  $U$  on both sides of (74) and employing Lemma 18, we find that

$$\begin{aligned} \sum_{n=-1}^{\infty} A_4(9n+10)q^n &= 2U\left(\frac{f_2^{12}}{q^3 f_1^6}\right) + 9\frac{f_4^8}{f_1^2} \\ &= 2\frac{f_2^6 f_3^{12}}{q f_1^6 f_6^6} + 40\frac{f_2^3 f_3^3 f_6^3}{f_1^3} + 2q\frac{f_6^{12}}{f_3^6} + 9\frac{f_4^8}{f_1^2}. \end{aligned} \quad (75)$$

*Proof (continued).* We now apply  $V$  on both sides of (75). In view of Theorem 16 and Lemma 19, we deduce that

$$\begin{aligned}
 \sum_{n=0}^{\infty} A_4(27n+19)q^n &= 2V\left(\frac{f_2^6 f_3^{12}}{q f_1^6 f_6^6}\right) + 40V\left(\frac{f_2^3 f_3^3 f_6^3}{f_1^3}\right) + 2\frac{f_2^{12}}{f_1^6} + 9\sum_{n=0}^{\infty} A_4(3n+1)q^n \\
 &= 2\left(21\frac{f_2^2 f_3^{10}}{f_1^4 f_6^2} + 48q\frac{f_3 f_6^7}{f_1 f_2}\right) + 40\left(3\frac{f_2^2 f_3^{10}}{f_1^4 f_6^2} + 3q\frac{f_3 f_6^7}{f_1 f_2}\right) \\
 &\quad + 2\frac{f_2^{12}}{f_1^6} + 9\left(2\frac{f_2^{12}}{f_1^6} + 9q^3\frac{f_{12}^8}{f_3^2}\right), \\
 &= 20\frac{f_2^{12}}{f_1^6} + 162\frac{f_2^2 f_3^{10}}{f_1^4 f_6^2} + 216q\frac{f_3 f_6^7}{f_1 f_2} + 81q^3\frac{f_{12}^8}{f_3^2},
 \end{aligned}$$

which yields the desired identity. □

### Theorem 21

For all integers  $n \geq 0$  with  $3 \nmid n$ , we have  $A_4(27n + 19) = 82A_4(3n + 1)$ . Consequently, for all integers  $n \geq 0$  with  $3 \nmid n$ , we have  $A_4(27n + 19) \equiv 0 \pmod{492}$ .

*Proof.* In view of (54) and Theorems 16 and 20, we consider the expansion of

$$\begin{aligned}
 & \sum_{n=0}^{\infty} (A_4(27n + 19) - 82A_4(3n + 1))q^n \\
 &= 18 \left( -8 \frac{f_2^{12}}{f_1^6} + 9 \frac{f_2^2 f_3^{10}}{f_1^4 f_6^2} + 12q \frac{f_3 f_6^7}{f_1 f_2} \right) - 657q^3 \frac{f_{12}^8}{f_3^2} \\
 &= 18 \left( \frac{f_1^3 f_2^3 f_6^3}{f_3^3} + 3q \frac{f_3 f_6^7}{f_1 f_2} \right) - 657q^3 \frac{f_{12}^8}{f_3^2}. \tag{76}
 \end{aligned}$$

*Proof (continued).* Utilizing (47) and (48), we see that

$$\begin{aligned}
 & \frac{f_1^3 f_2^3 f_6^3}{f_3^3} + 3q \frac{f_3 f_6^7}{f_1 f_2} \\
 &= \frac{f_6^3}{f_3^3} \cdot f_9^3 f_{18}^3 \left( \frac{1}{C(q^3)} - q - 2q^2 C(q^3) \right)^3 + 3q f_3 f_6^7 \cdot \frac{f_9^3 f_{18}^3}{f_3^4 f_6^4} \left( \frac{1}{C^2(q^3)} + \frac{q}{C(q^3)} \right. \\
 & \quad \left. + 3q^2 - 2q^3 C(q^3) + 4q^2 C^2(q^3) \right) \\
 &= \frac{f_6^3 f_9^3 f_{18}^3}{f_3^3} \left( \frac{1}{C^3(q^3)} + 20q^3 - 8q^6 C^3(q^3) \right). \tag{77}
 \end{aligned}$$

Observe that the right-hand side of (77) is a  $q$ -series whose terms are of the form  $q^{3n}$  only. Thus, the coefficients of the terms involving  $q^{3n+1}$  and  $q^{3n+2}$  in (76) are all zero, arriving at the desired identity. The second statement now follows from this identity and Theorem 17.  $\square$

*Proof of Theorem 3.* We proceed by induction on  $k$ . Appealing to Theorems 16 and 20, we see that (4) holds for  $k \in \{0, 1\}$ . Suppose now that (4) holds for some  $k \geq 2$ . Dividing both sides of (4) by  $q^3$  yields

$$\begin{aligned} & \sum_{n=-3}^{\infty} A_4 \left( 3^{2k+1}n + \frac{5(3^{2k+2} - 1)}{4} \right) q^n \\ &= \frac{3^{2k+2} - 1}{4} \cdot \frac{f_2^{12}}{q^3 f_1^6} + \frac{27(3^{2k} - 1)(3^{2k+2} - 1)}{320} \left( 3 \frac{f_2^2 f_3^{10}}{q^3 f_1^4 f_6^2} + 4 \frac{f_3 f_6^7}{q^2 f_1 f_2} \right) \\ &+ 3^{2k+2} \cdot \frac{f_{12}^8}{f_3^2}. \end{aligned} \tag{78}$$

*Proof (continued).* We apply  $U$  on both sides of (78) and use Lemma 18, obtaining

$$\begin{aligned}
 \sum_{n=-1}^{\infty} A_4 \left( 3^{2k+2}n + \frac{5(3^{2k+2} - 1)}{4} \right) q^n &= \frac{3^{2k+2} - 1}{4} U \left( \frac{f_2^{12}}{q^3 f_1^6} \right) \\
 &+ \frac{27(3^{2k} - 1)(3^{2k+2} - 1)}{320} \left[ 3U \left( \frac{f_2^2 f_3^{10}}{q^3 f_1^4 f_6^2} \right) + 4U \left( \frac{f_3 f_6^7}{q^2 f_1 f_2} \right) \right] \\
 &+ 3^{2k+2} \cdot \frac{f_4^8}{f_1^2} \\
 &= \frac{(3^{2k+2} - 1)(3^{2k+4} - 1)}{320} \left( \frac{f_2^6 f_3^{12}}{q f_1^6 f_6^6} + 20 \frac{f_2^3 f_3^3 f_6^3}{f_1^3} \right) + \frac{3^{2k+2} - 1}{4} \cdot \frac{q f_6^{12}}{f_3^6} \\
 &+ 3^{2k+2} \cdot \frac{f_4^8}{f_1^2}. \tag{79}
 \end{aligned}$$

*Proof (continued).* We now apply  $V$  on both sides of (79). In view of Theorem 16 and Lemma 19, we find that

$$\begin{aligned}
 & \sum_{n=0}^{\infty} A_4 \left( 3^{2k+3} n + \frac{3^{2k+4} - 5}{4} \right) q^n \\
 &= \frac{(3^{2k+2} - 1)(3^{2k+4} - 1)}{320} \left[ V \left( \frac{f_2^6 f_3^{12}}{q f_1^6 f_6^6} \right) + 20V \left( \frac{f_2^3 f_3^3 f_6^3}{f_1^3} \right) \right] \\
 &+ \frac{3^{2k+2} - 1}{4} \cdot \frac{f_2^{12}}{f_1^6} + 3^{2k+2} \sum_{n=0}^{\infty} A_4(3n+1)q^n \\
 &= \frac{3^{2k+4} - 1}{4} \cdot \frac{f_2^{12}}{f_1^6} + \frac{27(3^{2k+2} - 1)(3^{2k+4} - 1)}{320} \left( 3 \frac{f_2^2 f_3^{10}}{f_1^4 f_6^2} + 4q \frac{f_3 f_6^7}{f_1 f_2} \right) \\
 &+ 3^{2k+4} \cdot q^3 \frac{f_{12}^8}{f_3^2}. \tag{80}
 \end{aligned}$$

We infer from (43) that (4) also holds for  $k+1$ , so it holds for all  $k \geq 0$ .  $\square$

*Proof of Theorem 4.* For constants  $R_k$  and  $S_k$  with  $k \geq 0$ , we examine the expansion of

$$S := \sum_{n=0}^{\infty} \left[ A_4 \left( 3^{2k+1}n + \frac{3^{2k+2} - 5}{4} \right) - R_k A_4(27n + 19) - S_k A_4(3n + 1) \right] q^n.$$

Invoking Theorems 3, 16, and 20, we know that the above expression equals

$$\begin{aligned} S &= \left( \frac{3^{2k+2} - 1}{4} - 20R_k - 2S_k \right) \frac{f_2^{12}}{f_1^6} \\ &+ \left( \frac{27(3^{2k} - 1)(3^{2k+2} - 1)}{320} - 54R_k \right) \left( 3 \frac{f_2^2 f_3^{10}}{f_1^4 f_6^2} + 4q \frac{f_3 f_6^7}{f_1 f_2} \right) \\ &+ \left( 3^{2k+2} - 81R_k - 9S_k \right) \cdot q^3 \frac{f_{12}^8}{f_3^2}. \end{aligned} \tag{81}$$

*Proof (continued).* We now choose  $R_k$  and  $S_k$  so that the coefficients of

$$\frac{f_2^{12}}{f_1^6} \quad \text{and} \quad 3\frac{f_2^2 f_3^{10}}{f_1^4 f_6^2} + 4q\frac{f_3 f_6^7}{f_1 f_2}$$

are both zero for each  $k \geq 0$ . Thus, setting

$$\begin{aligned} \frac{3^{2k+2} - 1}{4} - 20R_k - 2S_k &= 0, \\ \frac{27(3^{2k} - 1)(3^{2k+2} - 1)}{320} - 54R_k &= 0, \end{aligned}$$

and solving this system of linear equations, we get

$$(R_k, S_k) = \left( \frac{(3^{2k} - 1)(3^{2k+2} - 1)}{640}, -\frac{(3^{2k} - 9)(3^{2k+2} - 1)}{64} \right).$$

*Proof (continued).* Thus, we deduce from (81) that

$$\begin{aligned} \sum_{n=0}^{\infty} A_4 \left( 3^{2k+1}n + \frac{3^{2k+2} - 5}{4} \right) q^n &= \frac{(3^{2k} - 1)(3^{2k+2} - 1)}{640} \sum_{n=0}^{\infty} A_4(27n + 19)q^n \\ &- \frac{(3^{2k} - 9)(3^{2k+2} - 1)}{64} \sum_{n=0}^{\infty} A_4(3n + 1)q^n + \frac{81(3^{2k} - 1)(3^{2k} - 9)}{640} \cdot q^3 \frac{f_{12}^8}{f_3^2}. \end{aligned} \quad (82)$$

Looking at the terms involving  $q^{3n+1}$  and  $q^{3n+2}$  in the expansion of (82), we find that

$$\begin{aligned} A_4 \left( 3^{2k+1}n + \frac{3^{2k+2} - 5}{4} \right) &= \frac{(3^{2k} - 1)(3^{2k+2} - 1)}{640} A_4(27n + 19) \\ &- \frac{(3^{2k} - 9)(3^{2k+2} - 1)}{64} A_4(3n + 1) \end{aligned} \quad (83)$$

for all integers  $n \geq 0$  with  $3 \nmid n$  and  $k \geq 0$ . Combining (83) and Theorem 21 yields (5). Hence, (6) now follows from (5) and Theorem 17.  $\square$

Thank you very much for your attention!!

(For references, please see [arXiv:2601.04743](https://arxiv.org/abs/2601.04743) and [arXiv:2601.10438](https://arxiv.org/abs/2601.10438).)