Partitions with parity restrictions: a bijective approach

Bruce Sagan
Michigan State University
www.math.msu.edu/~sagan

Joint work with William Keith

Seminar in Partion Theory, q-Series, and related topics
September 3, 2026

Outline

An *integer partition* of n, $\lambda \vdash n$, is a weakly decreasing sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_l)$ with $|\lambda| := \sum_i \lambda_i = n$. We display the *Young diagram* of λ in English notation with λ_i boxes in row i from the top. We use the notation $\lambda = \{\{1^{m_1}, 2^{m_2}, \ldots\}\}$ where m_i is the *multiplicity* of i as a part of λ. We will often combine notations and leave out commas and other delimiters in examples. The *conjugate* or *transpose* of λ is λ^t obtaned by reflecting the Young diagram of λ in the main diagonal.

Ex. We have

$$\lambda = (4, 4, 3, 1, 1) = \ydiagram{4,4,3,1,1} = \{\{1^2, 3, 4^2\}\}.$$

$$|\lambda| = 4 + 4 + 3 + 1 + 1 = 13, \text{ i.e., } \lambda \vdash 13 \qquad .$$

$$\lambda^t = \ydiagram{5,3,3,2} = (5, 3, 3, 2).$$

There has been recent interest in partitions with parity restrictions on their parts. Often these results have been proved algebraically. We give combinatorial proofs of theorems

1. by Andrews concerning parity-separated partitions,
2. by Chern, and by Passary concerning multiplicity restrictions by parity,
3. by Banerjee, Bringmann, and Dixit concerning parity-restricted overpartitions,
4. by Andrews concerning parity restrictions related to the third order mock theta function,
5. by Bringmann and Jennings-Shaffer concerning parity restrictions on distinct parts,
6. by Guadalupe concerning partition triples with parity restrictions,
7. by Matsakis and Vendervelde, and by Hemmer, Straub and Westrem concerning standard Young tableaux whose shapes are parity restricted.

Outline

- Introduction
- Parity separated partitions
- Adding multiplicity restrictions
- The third order mock theta function
- Open questions

We will use a hash tag for cardinality, upper case letters for sets, and the corresponding lower case letters for their cardinalities. Let

$$P_e^o(n) = \{\lambda \vdash n \mid \text{ every even part is smaller than every odd part}\},$$
$$p_e^o(n) = \# P_e^o(n).$$

Ex. $P_e^o(4) = \{4,\ 31,\ 2^2,\ 1^4\}$ so $p_e^o(4) = 4$.

Let

$$P_{e,1}(n) = \{\lambda \vdash n \mid \text{ every part is even or } 1\},$$
$$p_{e,1}(n) = \# P_{e,1}(n).$$

Ex. $P_{e,1}(4) = \{4,\ 2^2,\ 21^2,\ 1^4\}$ so $p_{e,1}(4) = 4$.

Theorem (Andrews)

For all $n \geq 0$ we have

$$p_e^o(n) = p_{e,1}(n).$$

Theorem (Andrews) For all $n \geq 0$ we have $p_e^o(n) = p_{e,1}(n)$.

Proof (Keith-S). We construct a bijection $f : P_e^o(n) \to P_{e,1}(n)$. If $\lambda = (\lambda_1, \ldots, \lambda_l)$ has odd parts $\lambda_1, \ldots, \lambda_k$ then let

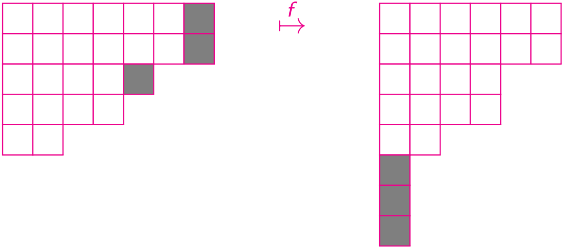
$$f(\lambda) = (\lambda_1 - 1, \ldots, \lambda_k - 1, \lambda_{k+1}, \ldots, \lambda_l, 1^k).$$

Note that $\mu = f(\lambda)$ is still a partition as

$$\lambda_1 - 1 \geq \ldots \geq \lambda_k - 1 \geq \lambda_{k+1} \geq \ldots \geq \lambda_l \geq 1 \geq \ldots \geq 1.$$

And $|\mu| = |\lambda| = n$. Also μ has only even parts and 1's by construction. To show f is a bijection, one constructs f^{-1}. $\square$

Ex.



Outline

Let

$$e(\lambda) = \text{largest even part of } \lambda, \text{ or } 0 \text{ if none exists.}$$
$$V_e^o(n) = \{\lambda \in \ P_e^o(n) \mid \ \text{all } m_i \text{ even, except } m_{e(\lambda)} \text{ is odd}\}.$$

Ex. $V_e^o(4) = \{4, \cancel{31}, \cancel{2^2}, 1^4\}$ so $v_e^o(4) = 2$.
Every part of $\lambda \in V_e^o(n)$ is either even or has even multiplicity (or both) so $V_e^o(n) \neq \emptyset$ imples n even. Let

$$V_e^o(n, m) = \{\lambda \in V_e^o(n) \mid e(\lambda) \equiv m \pmod 4\}.$$

Ex. $V_e^o(6, 0) = \{3^2, 1^6\}$ and $V_e^o(6, 2) = \{6, 2^3\}$ so $v_e^o(6, 0) = 2 = v_e^o(6, 2)$.

Theorem (Chern)

If $n \equiv 2 \pmod 4$ then $v_e^o(n, 0) = v_e^o(n, 2)$.

Chern's proof was algebraic. Burson and Eichhorn gave a demonstration using an iterated bijection.

Theorem (Keith-S)

If $n \equiv 2 \pmod 4$ then $\lambda \mapsto \lambda^t$ is a bijection $V_e^o(n, 0) \to V_e^o(n, 2)$.

Partition λ has *lattice path* $P(\lambda)$ obtained by walking along the boundary of the Young diagram lower left to upper right and recording E (east) or N (north) for each edge of a square traversed.

An *E-run* or *N-run* in $P(\lambda)$ is a maximal factor of the form E^i or N^j, respectively, where i, j are the *lengths* of the runs.

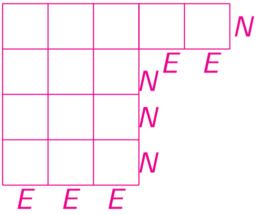
run N^j in $P(\lambda)$ $\leftrightarrow$ a part of multiplicity j in λ,

run E^i in $P(\lambda)$ $\leftrightarrow$ a part difference $\lambda_k - \lambda_{k+1} = i$ in λ.

The *transpose* of $P(\lambda)$ is

$P(\lambda)^t =$ read $P(\lambda)$ backwards and replace E with N, and N with E, $P(\lambda)^t = P(\lambda^t)$.

Ex.



$P(5,3,3,3) = E^3N^3E^2N$.

Four runs E^3, N^3, E^2, and N of lengths $3, 3, 2, 1$, respectively.

$P(\lambda)^t = EN^2E^3N^3$

Lemma (Keith-S)

We have $\lambda \in V_e^o(n)$ if and only if $P = P(\lambda)$ satisfies one of:

(a) *P has only 1 odd-length run which is the first or last run, or*

(b) *P has 2 odd-length runs which are consecutive in order $N^i E^j$.*

Proof sketch. Assume (a) (part (b) and the converse are similar.) If P has an initial odd-length run, it is an E-run and $\min \lambda$ is odd. All all other E-runs even means all parts of λ are odd. All N-runs are even, so all the parts have even multiplicity. So, $\lambda \in V_e^o(n)$. If P has a final odd-length run, then it is an N-run and $\max \lambda$ has odd multiplicity. Now all E-runs are even so all the parts of λ are even. Also, all N-runs except the last are even, so all parts of λ have even multiplicity except the largest part. So, $\lambda \in V_e^o(n)$. $\square$

Theorem (Keith-S)

If $n \equiv 2 \pmod 4$ then $\lambda \mapsto \lambda^t$ is a bijection $V_e^o(n, 0) \to V_e^o(n, 2)$.

Proof sketch. The lemma above and the description of $P(\lambda)^t = P(\lambda^t)$ show that $\lambda \mapsto \lambda^t$ restricts to a bijection on $V_e^o(n)$. Showing that it restricts to a bijection $V_e^o(n, 0) \to V_e^o(n, 2)$ is done by assuming $4 \mid e(\lambda)$ and $4 \mid e(\lambda^t)$ and showing $4|n$. $\rightarrow\leftarrow$

Outline

Introduction

Parity separated partitions

Adding multiplicity restrictions

The third order mock theta function

Open questions

The *Pochhammer symbol* and *third order mock theta function*

$$(a;q)_k = \prod_{i=1}^{k}(1-aq^{i-1}),$$

$$\nu(-q) = \sum_{k\geq 0}\frac{q^{k^2+k}}{(q;q^2)_{k+1}}$$
$$= \sum_{n\geq 0} p_\nu(n)q^n.$$

Let $A(n)$ be the set of $\lambda \vdash n$ such that

(A1) the even parts of λ are $2, 4, \ldots, e(\lambda)$ and all distinct,

(A2) any odd part of λ is at most $e(\lambda)+1$.

Proposition (Andrews)

For $n \geq 0$ we have

$$p_\nu(q) = \#A(n).$$

Proof. Numerator q^{k^2+k} counts partitions with parts $2, 4, \ldots, 2k$. And $1/(q;q^2)_{k+1}$ counts partitions into odd parts all at most $2k+1$. By (A1) and (A2), such pairs are counted by $\#A(n)$. $\square$

Let (i,j) be the cell in row i and column j of the Young diagram of λ. A *Young tableau* (YT), T, of shape λ is a filling of the cells of λ with positive integers. Let

$$T_{i,j} = \text{the integer in cell } (i,j) \text{ of } T,$$
$$|T| = \sum_{(i,j)\in\lambda} T_{i,j}.$$

The *odd Ferrers graph of λ* is the YT, T^λ, of shape λ with

$$T^\lambda_{i,j} = \begin{cases} 1 & \text{if } i = 1 \text{ or } j = 1, \\ 2 & \text{otherwise.} \end{cases}$$

Ex. The odd Ferrers graph of $\lambda = (6,4,4,3,1,1)$ is

$$T^\lambda = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 2 & 2 & 2 \\ \cline{1-4} 1 & 2 & 2 & 2 \\ \cline{1-4} 1 & 2 & 2 \\ \cline{1-3} 1 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array} \in B(13).$$

Let

$$B(n) = \{\, T^\lambda \mid \lambda \text{ self-conjugate and } |T^\lambda| = 2n+1 \}.$$

Theorem (Andrews)

For $n \geq 0$ we have

$$\#A(n) = \#B(n).$$

Proof (Keith-S). We define a bijection $g : B(n) \to A(n)$. Given $T^\lambda \in B(n)$ we let $\mu = g(T^\lambda)$ be the partition with parts

$$T^\lambda_{i,1} + T^\lambda_{i,2} + \cdots + T^\lambda_{i,i-1} + T^\lambda_{i,i}/2$$

for i from 2 to the number of parts of λ. Note that $T^\lambda_{i,j} = 0$ if $(i,j) \notin \lambda$. Verifying that $\mu \in A(n)$ and constructing g^{-1} are straight-forward. $\square$

Ex.

$$\begin{array}{rl} & \\ 1 + 2/2 & = 2 \\ 1 + 2 + 2/2 & = 4 \\ 1 + 2 + 2 + 0/2 & = 5 \\ 1 + 0 + 0 + 0 + 0/2 & = 1 \\ 1 + 0 + 0 + 0 + 0 + 0/2 & = 1 \end{array} \qquad \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 2 & 2 & 2 \\ \cline{1-4} 1 & 2 & 2 & 2 \\ \cline{1-4} 1 & 2 & 2 \\ \cline{1-3} 1 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array} \overset{g}{\mapsto} (5, 4, 2, 1, 1)$$

Outline

More on $V_e^o(n,m)$.

Theorem (Chern)

If $n \equiv 0 \pmod 4$ then

$$v_e^o(n,0) \geq v_e^o(n,2).$$

Burson and Eichhorn's interated bijection can be used to prove this. But when $n \equiv 0 \pmod 4$ we have $\lambda \in V_e^o(n,m)$ implies $\lambda^t \in V_e^o(n,m)$ for both values of m. Can some modification of transposition be used to prove this inequality?

Theorem (Chern)

We have

$$\sum_{n\geq 0} \left[v_e^o(n,0) - v_e^o(n,2)\right] q^n = \frac{(-q^4;q^4)_\infty}{(q^4;q^8)_\infty},$$

So, one could try to prove the inequality by finding a bijection

$$h : V_e^o(n,0) \to V_e^o(n,2) \uplus V(n)$$

where $\uplus$ is disjoint union and $V(n)$ is a set of (pairs of) partitions counted by $(-q^4;q^4)_\infty/(q^4;q^8)_\infty$.

More on the third order mock theta function.

Let

$$C(n) = \{\lambda \vdash 4n+1 \mid \lambda \text{ is self conjugate and all parts are odd}\}.$$

Ex. We have $C(2) = \{51^4,\ 3^3\}$ so $\#C(2) = 2$.

Let $D(n)$ be all $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$ with 0 allowed as a part and

(D1) $\lambda_1 > \lambda_2 > \ldots > \lambda_l \geq 0$, and

(D2) for all i: if λ_i is odd then $\lambda_i < 2\lambda_l$.

Ex. We have $D(2) = \{2,\ 20\}$ so $\#D(2) = 2$.

Theorem (Andrews)

For $n \geq 0$ we have

$$\#A(n) = \#B(n) = \#C(n) = \#D(n).$$

We have a simple bijection $B(n) \to C(n)$. We have been unable to find a bijection of $D(n)$ with any of the other sets. There are various parameters that such a bijection could preserve. For example, $\phi : A(n) \to D(n)$ such that the number of parts of $\phi(\lambda)$ is $1 + e(\lambda)/2$.

References

1. George E. Andrews. Integer partitions with even parts below odd parts and the mock theta functions. *Ann. Comb.*, 22(3):433–445, 2018.

2. Hannah E. Burson and Dennis Eichhorn. On the positivity of infinite products connected to partitions with even parts below odd parts and copartitions. *Ann. Comb.*, 29(1):197–210, 2025.

3. Shane Chern. Note on partitions with even parts below odd parts. *Math. Notes*, 110(3-4):454–457, 2021.

THANKS FOR LISTENING!