Identities of Hecke type and Rogers–Ramanujan type

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Notation

Let $|q| < 1$.

- q-Pochhammer symbols:

$$(\alpha; q)_n := \prod_{k=0}^{n-1} (1 - \alpha q^k),$$

$$(\alpha; q)_\infty := \prod_{k=0}^{\infty} (1 - \alpha q^k),$$

$$(\alpha, \beta, \ldots, \gamma; q)_n := (\alpha; q)_n (\beta; q)_n \cdots (\gamma; q)_n,$$

$$(\alpha, \beta, \ldots, \gamma; q)_\infty := (\alpha; q)_\infty (\beta; q)_\infty \cdots (\gamma; q)_\infty.$$

- q-Hypergeometric series ${}_{r+1}\phi_r$ with $|z| < 1$:

$${}_{r+1}\phi_r \left[\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_r \end{matrix} \,\middle|\, q; z \right] := \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_n}{(q, b_1, b_2, \ldots, b_r; q)_n} z^n.$$

Hecke type identities

Example (Rogers (1894), Hecke (1959)).

$$(q;q)_\infty^2 = \sum_{n=-\infty}^{\infty} \sum_{|m|\le n/2} (-1)^{m+n} q^{(n^2-3m^2)/2+(n+m)/2}.$$

Applications.

- Andrews (1986): Fifth- and seventh-order mock theta functions;
- Hickerson (1988): Mock theta conjectures;
- Hikami (2007): Unified Witten–Reshetikhin–Turaev invariants for certain manifolds.

Rogers–Ramanujan type identities

Example (Rogers (1894), Ramanujan (before 1913), R.–R. (1919), Schur (1917))**.**

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q,q^4;q^5)_\infty},$$
$$\sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}.$$

Applications.

- Lepowsky and Wilson (1978, 81, 84, 85): Affine Lie algebra;
- Baxter (1980): Solution of the hard hexagon model.

Motivation

Liuquan Wang, Ae Ja Yee (2020):

$$\sum_{n=1}^{\infty} \frac{q^n(q;q^2)_n}{(-q;q^2)_n(1+q^{2n})} = \sum_{n=1}^{\infty} \sum_{|m|\le n} (-1)^m q^{n^2+m^2} - \sum_{n=1}^{\infty} (-1)^n q^{2n^2}, \tag{1}$$

$$\sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q;q^2)_n(1+q^{2n})} = \sum_{n=1}^{\infty} \sum_{|m|\le n/2} (-1)^m q^{n^2-2m^2} - \sum_{n=1}^{\infty} (-1)^n q^{2n^2}, \tag{2}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^n(q^2;q^2)_{n-1}}{(-q^2;q^2)_n} = \sum_{n=1}^{\infty} \sum_{|m|\le n/2} (-1)^m q^{n^2-2m^2} - \sum_{n=1}^{\infty} (-1)^n q^{2n^2}. \tag{3}$$

Zhi-Guo Liu's transformation

Lemma

If A_n is a complex sequence, then, under suitable convergence condition, we have

$$\frac{(\alpha q, \alpha ab/q; q)_\infty}{(\alpha a, \alpha b; q)_\infty} \sum_{n=0}^{\infty} A_n (q/a; q)_n (\alpha a)^n$$
$$= \sum_{n=0}^{\infty} \frac{(1-\alpha q^{2n})(\alpha, q/a, q/b; q)_n (-\alpha ab/q)^n q^{n(n-1)/2}}{(1-\alpha)(q, \alpha a, \alpha b; q)_n}$$
$$\times \sum_{k=0}^{n} \frac{(q^{-n}, \alpha q^n; q)_k (q^2/b)^k}{(q/b; q)_k} A_k. \qquad (4)$$

Transformation A

$$\frac{(\alpha q, \alpha ab/q;q)_\infty}{(\alpha a, \alpha b;q)_\infty}\sum_{n=0}^{\infty} A_n (q/a;q)_n (\alpha a)^n$$
$$= \sum_{n=0}^{\infty} \frac{(1-\alpha q^{2n})(\alpha, q/a, q/b;q)_n (-\alpha ab/q)^n q^{n(n-1)/2}}{(1-\alpha)(q, \alpha a, \alpha b;q)_n} \sum_{k=0}^{n} \frac{(q^{-n}, \alpha q^n;q)_k (q^2/b)^k}{(q/b;q)_k} A_k.$$

Take

$$\alpha \to \alpha q \qquad \text{and} \qquad A_n = \frac{(q/b, \alpha cd/q, z;q)_n (b/q)^n}{(q, \alpha c, \alpha d, zq;q)_n}.$$

Then

$$\frac{(\alpha q, \alpha ab;q)_\infty}{(\alpha aq, \alpha bq;q)_\infty} {}_4\phi_3\left[\begin{matrix} q/a, q/b, \alpha cd/q, z \\ \alpha c, \alpha d, zq \end{matrix} \,\middle|\, q; \alpha ab\right]$$
$$= \sum_{n=0}^{\infty} \frac{(1-\alpha q^{2n+1})(\alpha q, q/a, q/b;q)_n (-\alpha ab)^n q^{n(n-1)/2}}{(q, \alpha aq, \alpha bq;q)_n}$$
$$\times {}_4\phi_3\left[\begin{matrix} q^{-n}, \alpha q^{n+1}, \alpha cd/q, z \\ \alpha c, \alpha d, zq \end{matrix} \,\middle|\, q; q\right]. \tag{5}$$

Transformation A

Replace b by q/z in the following special case of the Waston's q-analog of Whipple's theorem

$$\frac{(\alpha q, q^2/b; q)_n (b/q)^n}{(q, \alpha b; q)_n} {}_4\phi_3 \left[\begin{matrix} q^{-n}, \alpha q^{n+1}, q/b, \alpha cd/q \\ \alpha c, \alpha d, q^2/b \end{matrix} \middle| q; q \right]$$
$$= \sum_{j=0}^{n} \frac{(1-\alpha q^{2j})(\alpha, q/b, q/c, q/d; q)_j}{(1-\alpha)(q, \alpha b, \alpha c, \alpha d; q)_j} \left(\frac{\alpha bcd}{q^2} \right)^j. \tag{6}$$

Then

$${}_4\phi_3 \left[\begin{matrix} q^{-n}, \alpha q^{n+1}, \alpha cd/q, z \\ \alpha c, \alpha d, zq \end{matrix} \middle| q; q \right]$$
$$= \frac{(q, \alpha q/z; q)_n z^n}{(\alpha q, zq; q)_n} \sum_{j=0}^{n} \frac{(1-\alpha q^{2j})(\alpha, q/c, q/d, z; q)_j}{(1-\alpha)(q, \alpha c, \alpha d, \alpha q/z; q)_j} \left(\frac{\alpha cd}{zq} \right)^j. \tag{7}$$

Transformation A

Therefore,

Theorem

If α, a and b are complex numbers such that $|\alpha ab| < 1$, then

$$\frac{(\alpha q, \alpha ab; q)_\infty}{(\alpha aq, \alpha bq; q)_\infty} {}_4\phi_3 \left[\begin{matrix} q/a, q/b, \alpha cd/q, z \\ \alpha c, \alpha d, zq \end{matrix} \,\middle|\, q; \alpha ab \right]$$
$$= \sum_{n=0}^{\infty} \frac{(1-\alpha q^{2n+1})(q/a, q/b, \alpha q/z; q)_n (-\alpha abz)^n q^{n(n-1)/2}}{(\alpha aq, \alpha bq, zq; q)_n}$$
$$\times \sum_{j=0}^{n} \frac{(1-\alpha q^{2j})(\alpha, q/c, q/d, z; q)_j}{(1-\alpha)(q, \alpha c, \alpha d, \alpha q/z; q)_j} \left(\frac{\alpha cd}{zq} \right)^j. \tag{8}$$

Replace q by q^2. Let $\alpha \to 1$ and $(a, b, c, d, z) \to (1, q, -q, 0, -1)$. Then

$$\sum_{n=1}^{\infty} \frac{q^n (q; q^2)_n}{(-q; q^2)_n (1+q^{2n})} = \sum_{n=1}^{\infty} \sum_{j=-n+1}^{n} (-1)^j q^{n^2+j^2}.$$

Transformation A

Replace q by q^2. Let $\alpha \to 1$ and

$$(a,b,c,d,z) \to (1,q,-q,0,-1),\ (0,1,-q,\infty,-1),\ (1,q,-q^2,0,-q),$$
$$(0,1,-q^2,\infty,-q),\ (0,1,q,\infty,-q),\ (1,-q,-q^2,\infty,q).$$

Then

$$\sum_{n=1}^{\infty} \frac{q^n(q;q^2)_n}{(-q;q^2)_n(1+q^{2n})} = \sum_{n=1}^{\infty} \sum_{j=-n+1}^{n} (-1)^j q^{n^2+j^2},$$

$$\sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q;q^2)_n(1+q^{2n})} = \sum_{n=1}^{\infty} \sum_{j=-n+1}^{n} (-1)^{n+j} q^{2n^2-j^2},$$

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^n}{(-q^2;q^2)_n(1+q^{2n+1})} = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (-1)^j q^{n^2+n+j^2},$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q^2;q^2)_n(1+q^{2n+1})} = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (-1)^{n+j}(1-q^{2n+1}) q^{2n^2+n-j^2},$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-1)^n}{(q;q^2)_n(1+q^{2n+1})} = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (-1)^n(1-q^{2n+1}) q^{2n^2+n-j^2-j},$$

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^n}{(-q^2;q^2)_n(1-q^{2n+1})} = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} q^{n^2+n-j^2}.$$

Transformation B

$$\frac{(\alpha q, \alpha ab/q; q)_\infty}{(\alpha a, \alpha b; q)_\infty} \sum_{n=0}^{\infty} A_n (q/a; q)_n (\alpha a)^n$$
$$= \sum_{n=0}^{\infty} \frac{(1-\alpha q^{2n})(\alpha, q/a, q/b; q)_n (-\alpha ab/q)^n q^{n(n-1)/2}}{(1-\alpha)(q, \alpha a, \alpha b; q)_n} \sum_{k=0}^{n} \frac{(q^{-n}, \alpha q^n; q)_k (q^2/b)^k}{(q/b; q)_k} A_k.$$

Replace q by q^2 and take

$$\alpha \to \alpha^2 q^2 \qquad \text{and} \qquad A_n = \frac{(q^2/b, \lambda, \lambda q, z; q^2)_n (b/q^2)^n}{(q^2, \alpha q, \alpha q^2, \lambda^2, z q^2; q^2)_n}.$$

Then

$$\frac{(\alpha^2 q^2, \alpha^2 ab; q^2)_\infty}{(\alpha^2 a q^2, \alpha^2 b q^2; q^2)_\infty} {}_5\phi_4 \left[\begin{matrix} q^2/a, q^2/b, \lambda, \lambda q, z \\ \alpha q, \alpha q^2, \lambda^2, z q^2 \end{matrix} \,\middle|\, q^2; \alpha^2 ab \right]$$
$$= \sum_{n=0}^{\infty} \frac{(1-\alpha^2 q^{4n+2})(\alpha^2 q^2, q^2/a, q^2/b; q^2)_n (-\alpha^2 ab)^n q^{n^2-n}}{(q^2, \alpha^2 a q^2, \alpha^2 b q^2; q^2)_n}$$
$$\times {}_5\phi_4 \left[\begin{matrix} q^{-2n}, \alpha^2 q^{2n+2}, \lambda, q\lambda, z \\ \alpha q, \alpha q^2, \lambda^2, z q^2 \end{matrix} \,\middle|\, q^2; q^2 \right]. \tag{9}$$

Transformation B

Take $(a,b)\to(\alpha^{-2}q^{-2n},z^{-1}q^2)$ in a result due to Z.-G. Liu

$$\frac{(\alpha^2q^2,\alpha^2ab/q^2;q^2)_n}{(\alpha^2a,\alpha^2b;q^2)_n}{}_5\phi_4\left[\begin{matrix}q^{-2n},q^2/a,q^2/b,\lambda,\lambda q\\ \alpha q,\alpha q^2,\lambda^2,q^4/\alpha^2abq^{2n}\end{matrix}\;\middle|\;q^2;q^2\right]$$
$$=\sum_{j=0}^{n}\frac{(1+\alpha q^{2j})(q^{-2n},\alpha^2,q^2/a,q^2/b;q^2)_j(-q,q\alpha/\lambda;q)_j(\alpha^2ab\lambda q^{2n-2})^j}{(1+\alpha)(q^2,\alpha^2q^{2n+2},\alpha^2a,\alpha^2b;q^2)_j(\alpha,-\lambda;q)_j} \tag{10}$$

Then

$${}_5\phi_4\left[\begin{matrix}q^{-2n},\alpha^2q^{2n+2},\lambda,\lambda q,z\\ \alpha q,\alpha q^2,\lambda^2,zq^2\end{matrix}\;\middle|\;q^2;q^2\right]$$
$$=\frac{(q^2,\alpha^2q^2/z;q^2)_nz^n}{(\alpha^2q^2,zq^2;q^2)_n}\sum_{j=0}^{n}\frac{(1+\alpha q^{2j})(\alpha^2,z;q^2)_j(-q,q\alpha/\lambda;q)_j(\lambda/z)^j}{(1+\alpha)(q^2,\alpha^2q^2/z;q^2)_j(\alpha,-\lambda;q)_j}. \tag{11}$$

Transformation B

Therefore,

Theorem

If α, a and b are complex numbers such that $|\alpha^2 ab| < 1$, then

$$\frac{(\alpha^2 q^2, \alpha^2 ab; q^2)_\infty}{(\alpha^2 aq^2, \alpha^2 bq^2; q^2)_\infty} {}_5\phi_4\left[\begin{matrix} q^2/a, q^2/b, \lambda, \lambda q, z \\ \alpha q, \alpha q^2, \lambda^2, zq^2 \end{matrix} \,\middle|\, q^2; \alpha^2 ab\right]$$
$$= \sum_{n=0}^{\infty} \frac{(1-\alpha^2 q^{4n+2})(q^2/a, q^2/b, \alpha^2 q^2/z; q^2)_n (-\alpha^2 abz)^n q^{n^2-n}}{(\alpha^2 aq^2, \alpha^2 bq^2, zq^2; q^2)_n}$$
$$\times \sum_{j=0}^{n} \frac{(1+\alpha q^{2j})(\alpha^2, z; q^2)_j (-q, \alpha q/\lambda; q)_j (\lambda/z)^j}{(1+\alpha)(q^2, \alpha^2 q^2/z; q^2)_j (\alpha, -\lambda; q)_j}. \tag{12}$$

Transformation B

Let $\alpha \to -1$ and

$$(a, b, \lambda, z) \to (1, q, q, -q),\ (1, q, 0, -q),\ (1, q, \infty, -q),\ (1, q, q, 0),$$
$$(1, q, q^{1/2}, 0),\ (1, q, q^{1/2}, -q).$$

Then

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n^2 q^n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{n^2+n},$$

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{3n(n+1)/2},$$

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n (-1)^n q^{n^2+n}}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2},$$

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n^2 q^n}{(-q;q)_{2n}} = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (-1)^j (1+q^{2n+1}) q^{2n^2+n-j^2},$$

$$\sum_{n=0}^{\infty} \frac{(q^{1/2};q)_{2n} q^n}{(-q;q)_{2n}} = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (-1)^j (1+q^{2n+1}) q^{2n^2+n-j^2+j/2},$$

$$\sum_{n=0}^{\infty} \frac{(q^{1/2};q)_{2n} q^n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (-1)^j q^{n^2+n+j/2}.$$

Transformation C

$$\frac{(\alpha q, \alpha ab/q; q)_\infty}{(\alpha a, \alpha b; q)_\infty} \sum_{n=0}^{\infty} A_n (q/a; q)_n (\alpha a)^n$$
$$= \sum_{n=0}^{\infty} \frac{(1-\alpha q^{2n})(\alpha, q/a, q/b; q)_n (-\alpha ab/q)^n q^{n(n-1)/2}}{(1-\alpha)(q, \alpha a, \alpha b; q)_n} \sum_{k=0}^{n} \frac{(q^{-n}, \alpha q^n; q)_k (q^2/b)^k}{(q/b; q)_k} A_k.$$

Take

$$\alpha \to \alpha q \quad \text{and} \quad A_n = \frac{(q/b, \alpha^{1/3} q^{1/3}, \alpha^{1/3} q^{2/3}, \alpha^{1/3} q, z; q)_n (b/q)^n}{(q, \alpha^{1/2} q, -\alpha^{1/2} q, \alpha^{1/2} q^{1/2}, -\alpha^{1/2} q^{1/2}, zq; q)_n}.$$

Then

$$\frac{(\alpha q, \alpha ab; q)_\infty}{(\alpha aq, \alpha bq; q)_\infty} {}_6\phi_5 \left[\begin{matrix} q/a, q/b, \alpha^{1/3} q^{1/3}, \alpha^{1/3} q^{2/3}, \alpha^{1/3} q, z \\ \alpha^{1/2} q, -\alpha^{1/2} q, \alpha^{1/2} q^{1/2}, -\alpha^{1/2} q^{1/2}, zq \end{matrix} \,\middle|\, q; \alpha ab \right]$$
$$= \sum_{n=0}^{\infty} \frac{(1-\alpha q^{2n+1})(\alpha q, q/a, q/b; q)_n (-\alpha ab)^n q^{n(n-1)/2}}{(q, \alpha aq, \alpha bq; q)_n}$$
$$\times \; {}_6\phi_5 \left[\begin{matrix} q^{-n}, \alpha q^{n+1}, \alpha^{1/3} q^{1/3}, \alpha^{1/3} q^{2/3}, \alpha^{1/3} q, z \\ \alpha^{1/2} q, -\alpha^{1/2} q, \alpha^{1/2} q^{1/2}, -\alpha^{1/2} q^{1/2}, zq \end{matrix} \,\middle|\, q; q \right]. \tag{13}$$

Transformation C

Theorem

For each non-negative integer n, we have

$$\frac{(\alpha q, \alpha ab/q; q)_n}{(\alpha a, \alpha b; q)_n} {}_6\phi_5 \left[\begin{matrix} q^{-n}, q/a, q/b, \alpha^{1/3}q^{1/3}, \alpha^{1/3}q^{2/3}, \alpha^{1/3}q \\ \alpha^{1/2}q, -\alpha^{1/2}q, \alpha^{1/2}q^{1/2}, -\alpha^{1/2}q^{1/2}, q^2/\alpha abq^n \end{matrix} \middle| q; q \right]$$
$$= \sum_{j=0}^{n} \frac{(1-\alpha^{1/3}q^{2j/3})(q^{-n}, q/a, q/b; q)_j (\alpha^{1/3}; q^{1/3})_j (\alpha abq^{n-1})^j (\alpha q)^{j/3}}{(1-\alpha^{1/3})(\alpha q^{n+1}, \alpha a, \alpha b; q)_j (q^{1/3}; q^{1/3})_j}. \tag{14}$$

Proof. Recall another general transformation of Z.-G. Liu

$$\frac{(\alpha q, \alpha ab/q; q)_m}{(\alpha a, \alpha b; q)_m} \sum_{n=0}^{m} \frac{(q^{-m}, q/a, q/b; q)_n q^n}{(q^2/\alpha abq^m; q)_n} A_n$$
$$= \sum_{n=0}^{m} \frac{(1-\alpha q^{2n})(q^{-m}, \alpha, q/a, q/b; q)_n (\alpha abq^{m-1})^n}{(1-\alpha)(q, \alpha q^{m+1}, \alpha a, \alpha b; q)_n} \sum_{k=0}^{n} (q^{-n}, \alpha q^n; q)_k q^k A_k. \tag{15}$$

Take

$$A_n = \frac{(\alpha^{1/3}q^{1/3}, \alpha^{1/3}q^{2/3}, \alpha^{1/3}q; q)_n}{(q, \alpha^{1/2}q, -\alpha^{1/2}q, \alpha^{1/2}q^{1/2}, -\alpha^{1/2}q^{1/2}; q)_n}.$$

Transformation C

Then

$$\frac{(\alpha q, \alpha ab/q; q)_m}{(\alpha a, \alpha b; q)_m} {}_6\phi_5 \left[\begin{matrix} q^{-m}, q/a, q/b, \alpha^{1/3}q^{1/3}, \alpha^{1/3}q^{2/3}, \alpha^{1/3}q \\ \alpha^{1/2}q, -\alpha^{1/2}q, \alpha^{1/2}q^{1/2}, -\alpha^{1/2}q^{1/2}, q^2/\alpha abq^m \end{matrix} \;\middle|\; q; q \right]$$

$$= \sum_{n=0}^{m} \frac{(1-\alpha q^{2n})(q^{-m}, \alpha, q/a, q/b; q)_n (\alpha abq^{m-1})^n}{(1-\alpha)(q, \alpha q^{m+1}, \alpha a, \alpha b; q)_n}$$

$$\times {}_5\phi_4 \left[\begin{matrix} q^{-n}, \alpha q^n, \alpha^{1/3}q^{1/3}, \alpha^{1/3}q^{2/3}, \alpha^{1/3}q \\ \alpha^{1/2}q, -\alpha^{1/2}q, \alpha^{1/2}q^{1/2}, -\alpha^{1/2}q^{1/2} \end{matrix} \;\middle|\; q; q \right]. \tag{16}$$

Finally, we apply an identity of Andrews

$${}_5\phi_4 \left[\begin{matrix} q^{-n}, \alpha q^n, \alpha^{1/3}q^{1/3}, \alpha^{1/3}q^{2/3}, \alpha^{1/3}q \\ \alpha^{1/2}q, -\alpha^{1/2}q, \alpha^{1/2}q^{1/2}, -\alpha^{1/2}q^{1/2} \end{matrix} \;\middle|\; q; q \right]$$

$$= \frac{(1-\alpha)(1-\alpha^{1/3}q^{2n/3})(q;q)_n(\alpha^{1/3};q^{1/3})_n(q\alpha)^{n/3}}{(1-\alpha^{1/3})(1-\alpha q^{2n})(\alpha;q)_n(q^{1/3};q^{1/3})_n}. \tag{17}$$

□

Transformation C

$$\frac{(\alpha q, \alpha ab/q; q)_n}{(\alpha a, \alpha b; q)_n} {}_6\phi_5\left[\begin{matrix} q^{-n}, q/a, q/b, \alpha^{1/3}q^{1/3}, \alpha^{1/3}q^{2/3}, \alpha^{1/3}q \\ \alpha^{1/2}q, -\alpha^{1/2}q, \alpha^{1/2}q^{1/2}, -\alpha^{1/2}q^{1/2}, q^2/\alpha abq^n \end{matrix} \middle| q; q\right]$$
$$= \sum_{j=0}^{n} \frac{(1-\alpha^{1/3}q^{2j/3})(q^{-n}, q/a, q/b; q)_j(\alpha^{1/3}; q^{1/3})_j(\alpha abq^{n-1})^j(\alpha q)^{j/3}}{(1-\alpha^{1/3})(\alpha q^{n+1}, \alpha a, \alpha b; q)_j(q^{1/3}; q^{1/3})_j}.$$

Take $(a, b) \to (\alpha^{-1}q^{-n}, z^{-1}q)$. Then

$${}_6\phi_5\left[\begin{matrix} q^{-n}, \alpha q^{n+1}, \alpha^{1/3}q^{1/3}, \alpha^{1/3}q^{2/3}, \alpha^{1/3}q, z \\ \alpha^{1/2}q, -\alpha^{1/2}q, \alpha^{1/2}q^{1/2}, -\alpha^{1/2}q^{1/2}, zq \end{matrix} \middle| q; q\right]$$
$$= \frac{(q, \alpha q/z; q)_n z^n}{(\alpha q, zq; q)_n} \sum_{j=0}^{n} \frac{(1-\alpha^{1/3}q^{2j/3})(z; q)_j(\alpha^{1/3}; q^{1/3})_j z^{-j}(q\alpha)^{j/3}}{(1-\alpha^{1/3})(\alpha q/z; q)_j(q^{1/3}; q^{1/3})_j}. \quad (18)$$

Transformation C

Therefore,

Theorem

If α, a and b are complex numbers such that $|\alpha ab| < 1$, then

$$\frac{(\alpha q, \alpha ab; q)_\infty}{(\alpha aq, \alpha bq; q)_\infty} {}_6\phi_5 \left[\begin{matrix} q/a, q/b, \alpha^{1/3}q^{1/3}, \alpha^{1/3}q^{2/3}, \alpha^{1/3}q, z \\ \alpha^{1/2}q, -\alpha^{1/2}q, \alpha^{1/2}q^{1/2}, -\alpha^{1/2}q^{1/2}, zq \end{matrix} \middle| q; \alpha ab \right]$$
$$= \sum_{n=0}^{\infty} \frac{(1-\alpha q^{2n+1})(q/a, q/b, \alpha q/z; q)_n (-\alpha abz)^n q^{n(n-1)/2}}{(\alpha aq, \alpha bq, zq; q)_n}$$
$$\times \sum_{j=0}^{n} \frac{(1-\alpha^{1/3}q^{2j/3})(z;q)_j (\alpha^{1/3}; q^{1/3})_j z^{-j} (q\alpha)^{j/3}}{(1-\alpha^{1/3})(\alpha q/z; q)_j (q^{1/3}; q^{1/3})_j}. \tag{19}$$

Replace q by q^6. Let $(\alpha, a, b, z) \to (1, 1, q^3, 0)$. Then

$$\sum_{n=0}^{\infty} \frac{(q^2;q^2)_{3n}(q^3;q^6)_n q^{3n}}{(q^6;q^6)_{2n}} = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (1+q^{6n+3})(-1)^j q^{6n^2+3n-3j^2-j}.$$

Transformation C*

Recall (14)

$$\frac{(\alpha q, \alpha ab/q; q)_n}{(\alpha a, \alpha b; q)_n} {}_6\phi_5 \left[\begin{matrix} q^{-n}, q/a, q/b, \alpha^{1/3}q^{1/3}, \alpha^{1/3}q^{2/3}, \alpha^{1/3}q \\ \alpha^{1/2}q, -\alpha^{1/2}q, \alpha^{1/2}q^{1/2}, -\alpha^{1/2}q^{1/2}, q^2/\alpha abq^n \end{matrix} \middle| q; q \right]$$

$$= \sum_{j=0}^{n} \frac{(1-\alpha^{1/3}q^{2j/3})(q^{-n}, q/a, q/b; q)_j (\alpha^{1/3}; q^{1/3})_j (\alpha abq^{n-1})^j (\alpha q)^{j/3}}{(1-\alpha^{1/3})(\alpha q^{n+1}, \alpha a, \alpha b; q)_j (q^{1/3}; q^{1/3})_j}.$$

Let $n \to \infty$ and $(\alpha, q) \to (\alpha^3, q^3)$. Then

Theorem

If α, a and b are complex numbers such that $|\alpha^3 ab/q^3| < 1$, then

$$\frac{(\alpha^3 q^3, \alpha^3 ab/q^3; q^3)_\infty}{(\alpha^3 a, \alpha^3 b; q^3)_\infty} {}_5\phi_4 \left[\begin{matrix} q^3/a, q^3/b, \alpha q, \alpha q^2, \alpha q^3 \\ \alpha^{3/2}q^3, -\alpha^{3/2}q^3, \alpha^{3/2}q^{3/2}, -\alpha^{3/2}q^{3/2} \end{matrix} \middle| q^3; \frac{\alpha^3 ab}{q^3} \right]$$

$$= \sum_{j=0}^{\infty} \frac{(1-\alpha q^{2j})(\alpha; q)_j (q^3/a, q^3/b; q^3)_j (-\alpha^4 ab)^j q^{(3j^2-7j)/2}}{(1-\alpha)(q; q)_j (\alpha^3 a, \alpha^3 b; q^3)_j}. \quad (20)$$

Transformation C*

$$\frac{(\alpha^3 q^3, \alpha^3 ab/q^3; q^3)_\infty}{(\alpha^3 a, \alpha^3 b; q^3)_\infty} {}_5\phi_4 \left[\begin{matrix} q^3/a, q^3/b, \alpha q, \alpha q^2, \alpha q^3 \\ \alpha^{3/2} q^3, -\alpha^{3/2} q^3, \alpha^{3/2} q^{3/2}, -\alpha^{3/2} q^{3/2} \end{matrix} \,\Big|\, q^3; \frac{\alpha^3 ab}{q^3} \right]$$
$$= \sum_{j=0}^{\infty} \frac{(1-\alpha q^{2j})(\alpha; q)_j (q^3/a, q^3/b; q^3)_j (-\alpha^4 ab)^j q^{(3j^2-7j)/2}}{(1-\alpha)(q;q)_j (\alpha^3 a, \alpha^3 b; q^3)_j}.$$

Let

$$(\alpha, a, b, z) \to (q, 0, 0, 0),\ (1, 0, 0, 0).$$

Then

$$\sum_{n=0}^{\infty} \frac{q^{3n^2+3n}(q;q)_{3n+1}}{(q^3;q^3)_n (q^3;q^3)_{2n+1}} = \frac{1}{(q^3;q^3)_\infty} \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{9j^2+7j}{2}} = \frac{(q, q^8, q^9; q^9)_\infty}{(q^3;q^3)_\infty},$$
$$\sum_{n=0}^{\infty} \frac{q^{3n^2}(q;q)_{3n}}{(q^3;q^3)_n (q^3;q^3)_{2n}} = \frac{1}{(q^3;q^3)_\infty} \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{9j^2-j}{2}} = \frac{(q^4, q^5, q^9; q^9)_\infty}{(q^3;q^3)_\infty}.$$

A general transformation

Theorem

Let A_n be a complex sequence. Then, under suitable convergence conditions, we have

$$\sum_{m=0}^{\infty} \frac{(a/x, y/x; q)_m}{(q, aq/y; q)_m} \left(\frac{x^2q}{y}\right)^m \sum_{n=0}^{m} \frac{(q^{-m}, q^m a/x, y; q)_n}{(y/x; q)_n} q^n A_n$$
$$= \frac{(axq, xq; q)_\infty}{(aq, x^2q; q)_\infty} \sum_{m=0}^{\infty} \frac{(1-aq^{2m})(a, y; q)_m (a/x; q)_{2m}}{(1-a)(q, aq/y; q)_m (axq; q)_{2m}} \left(\frac{x^2q}{y}\right)^m$$
$$\times \sum_{n=0}^{m} (q^{-m}, aq^m; q)_n q^n A_n. \tag{21}$$

A general transformation

Recall (15)

$$\frac{(\alpha q,\alpha ab/q;q)_m}{(\alpha a,\alpha b;q)_m}\sum_{n=0}^{m}\frac{(q^{-m},q/a,q/b;q)_n q^n}{(q^2/\alpha abq^m;q)_n}A_n$$
$$=\sum_{n=0}^{m}\frac{(1-\alpha q^{2n})(q^{-m},\alpha,q/a,q/b;q)_n(\alpha abq^{m-1})^n}{(1-\alpha)(q,\alpha q^{m+1},\alpha a,\alpha b;q)_n}\sum_{k=0}^{n}(q^{-n},\alpha q^n;q)_k q^k A_k.$$

Proof. Take $(a,b,\alpha)=(xq^{1-m}/a,q/y,a)$. Then

$$\frac{(aq,y/x;q)_m}{(1/x,aq/y;q)_m y^m}\sum_{n=0}^{m}\frac{(q^{-m},aq^m/x,y;q)_n}{(y/x;q)_n}q^n A_n$$
$$=\sum_{n=0}^{m}\frac{(1-aq^{2n})(a,aq^m/x,y,q^{m-n+1};q)_n}{(1-a)(q,aq^{m+1},q^{m-n}/x,aq/y;q)_n y^n}\sum_{k=0}^{n}(q^{-n},aq^n;q)_k q^k A_k.$$

A general transformation

We next multiply both sides of the above identity by

$$\frac{(a/x, 1/x; q)_m (x^2 q)^m}{(q, aq; q)_m}.$$

Summing m over nonnegative integers then yields

$$\sum_{m=0}^{\infty} \frac{(a/x, y/x; q)_m}{(q, aq/y; q)_m} \left(\frac{x^2 q}{y}\right)^m \sum_{n=0}^{m} \frac{(q^{-m}, q^m a/x, y; q)_n}{(y/x; q)_n} q^n A_n$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(1 - aq^{2n})(a, y; q)_n (1/x; q)_{m-n}}{(1 - a)(q, aq/y; q)_n (q; q)_{m-n}}$$

$$\times \frac{(a/x; q)_{m+n} (x^2 q)^m}{(aq; q)_{m+n} y^n} \sum_{k=0}^{n} (q^{-n}, aq^n; q)_k q^k A_k.$$

A general transformation

It follows by interchanging the first two summations on the right-hand side that

$$\sum_{m=0}^{\infty}\frac{(a/x,y/x;q)_m}{(q,aq/y;q)_m}\left(\frac{x^2q}{y}\right)^m\sum_{n=0}^{m}\frac{(q^{-m},q^m a/x,y;q)_n}{(y/x;q)_n}q^nA_n$$
$$=\sum_{n=0}^{\infty}\frac{(1-aq^{2n})(a,y;q)_n(a/x;q)_{2n}}{(1-a)(q,aq/y;q)_n(aq;q)_{2n}}\left(\frac{x^2q}{y}\right)^n$$
$$\times\sum_{m=0}^{\infty}\frac{(1/x;q)_m(aq^{2n}/x;q)_m(x^2q)^m}{(q;q)_m(aq^{2n+1};q)_m}\sum_{k=0}^{n}(q^{-n},aq^n;q)_kq^kA_k.$$

Finally, for the second summation on the right-hand side, we apply the q-Gauß summation:

$${}_2\phi_1\left[\begin{matrix}a,b\\c\end{matrix}\,\middle|\,q;\frac{c}{ab}\right]=\frac{(c/a,c/b;q)_\infty}{(c,c/ab;q)_\infty}.$$

□

Transformation D

$$\sum_{m=0}^{\infty}\frac{(a/x,y/x;q)_m}{(q,aq/y;q)_m}\left(\frac{x^2q}{y}\right)^m\sum_{n=0}^{m}\frac{(q^{-m},q^ma/x,y;q)_n}{(y/x;q)_n}q^nA_n$$
$$=\frac{(axq,xq;q)_\infty}{(aq,x^2q;q)_\infty}\sum_{m=0}^{\infty}\frac{(1-aq^{2m})(a,y;q)_m(a/x;q)_{2m}}{(1-a)(q,aq/y;q)_m(axq;q)_{2m}}\left(\frac{x^2q}{y}\right)^m\sum_{n=0}^{m}(q^{-m},aq^m;q)_nq^nA_n.$$

Take

$$A_n=\frac{(\sqrt{\lambda},-\sqrt{\lambda};q)_n}{(q,\sqrt{aq},-\sqrt{aq},\lambda;q)_n}.$$

Then

$$\sum_{m=0}^{\infty}\frac{(a/x,y/x;q)_m}{(q,aq/y;q)_m}\left(\frac{x^2q}{y}\right)^m{}_5\phi_4\left[\begin{matrix}q^{-m},q^ma/x,y,\sqrt{\lambda},-\sqrt{\lambda}\\ \sqrt{aq},-\sqrt{aq},\lambda,y/x\end{matrix}\,\middle|\,q;q\right]$$
$$=\frac{(axq,xq;q)_\infty}{(aq,x^2q;q)_\infty}\sum_{m=0}^{\infty}\frac{(1-aq^{2m})(a,y;q)_m(a/x;q)_{2m}}{(1-a)(q,aq/y;q)_m(axq;q)_{2m}}\left(\frac{x^2q}{y}\right)^m$$
$$\times{}_4\phi_3\left[\begin{matrix}q^{-m},q^ma,\sqrt{\lambda},-\sqrt{\lambda}\\ \sqrt{aq},-\sqrt{aq},\lambda\end{matrix}\,\middle|\,q;q\right].$$

Transformation D

Apply an identity due to Andrews:

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, aq^n, \sqrt{\lambda}, -\sqrt{\lambda} \\ \sqrt{aq}, -\sqrt{aq}, \lambda \end{matrix} \,\middle|\, q;q\right] = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \dfrac{(q, aq/\lambda; q^2)_{n/2}(\lambda)^{n/2}}{(aq, \lambda q; q^2)_{n/2}} & \text{if } n \text{ is even,} \end{cases} $$

Theorem

$$ \sum_{m=0}^{\infty} \frac{(a/x, y/x; q)_m}{(q, aq/y; q)_m} \left(\frac{x^2 q}{y}\right)^m {}_5\phi_4\left[\begin{matrix} q^{-m}, q^m a/x, y, \sqrt{\lambda}, -\sqrt{\lambda} \\ \sqrt{aq}, -\sqrt{aq}, \lambda, y/x \end{matrix} \,\middle|\, q;q\right] $$
$$ = \frac{(axq, xq; q)_\infty}{(aq, x^2 q; q)_\infty} \sum_{m=0}^{\infty} \frac{(1-aq^{4m})(a, aq/\lambda; q^2)_m (y;q)_{2m} (a/x;q)_{4m}}{(1-a)(q^2, q\lambda; q^2)_m (aq/y; q)_{2m} (axq;q)_{4m}} \left(\frac{x^4 q^2 \lambda}{y^2}\right)^m . \tag{22} $$

Transformation D

Let $x \to 0$, $y \to \infty$ and $\lambda = -a^{1/2}$ in (22). Then

$$\sum_{s,t\geq 0} \frac{(-a^{1/2};q^2)_s a^{2s+t} q^{2s^2+t^2+2st}}{(q;q)_t(q;q)_s(aq;q^2)_s(-a^{1/2};q)_s}$$
$$= \frac{1}{(aq;q)_\infty} \sum_{m=0}^{\infty} \frac{(1-aq^{4m})(a;q^2)_m}{(1-a)(q^2;q^2)_m} (-1)^m a^{\frac{9m}{2}} q^{10m^2-m}.$$

Take $a=1$ and $a=q^2$ and apply JTP,

$$\sum_{s,t\geq 0} \frac{(-1;q^2)_s q^{2s^2+t^2+2st}}{(q;q)_t(q;q)_s(q;q^2)_s(-1;q)_s} = \frac{(q^9,q^{11},q^{20};q^{20})_\infty}{(q;q)_\infty}, \tag{23}$$

$$\sum_{s,t\geq 0} \frac{(-q;q^2)_s q^{2s^2+t^2+2st+4s+2t}}{(q;q)_t(q;q)_{2s+1}} = \frac{(q^2,q^{18},q^{20};q^{20})_\infty}{(q;q)_\infty}. \tag{24}$$

Transformation D

Let $x \to 0$, $y = (aq)^{1/2}$ and $\lambda = -a^{1/2}$ in (22). Then

$$\sum_{s,t\geq 0} \frac{(-a^{1/2};q^2)_s(-1)^s a^{\frac{3}{2}s+t} q^{\frac{3}{2}s^2+t^2+2st}}{(q;q)_t(q;q)_s(aq;q^2)_s(-a^{1/2};q)_s(a^{1/2}q^{s+1/2};q)_t}$$

$$= \frac{1}{(aq;q)_\infty} \sum_{m=0}^{\infty} \frac{(1-aq^{4m})(a;q^2)_m}{(1-a)(q^2;q^2)_m} (-1)^m a^{\frac{7m}{2}} q^{8m^2-m}.$$

Take $a = 1$ and $a = q^2$ and apply JTP,

$$\sum_{s,t\geq 0} \frac{(-1;q^2)_s(-1)^s q^{\frac{3}{2}s^2+t^2+2st}}{(q;q)_t(q;q)_s(q;q^2)_s(-1;q)_s(q^{1/2+s};q)_t} = \frac{(q^7,q^9,q^{16};q^{16})_\infty}{(q;q)_\infty}, \tag{25}$$

$$\sum_{s,t\geq 0} \frac{(-q;q^2)_s(-1)^s q^{\frac{3}{2}s^2+t^2+2st+3s+2t}}{(q;q)_t(q;q)_{2s+1}(q^{3/2+s};q)_t} = \frac{(q^2,q^{14},q^{16};q^{16})_\infty}{(q;q)_\infty}. \tag{26}$$

Transformation E

Define

$$M_i := \begin{cases} 0 & \text{if } i = 0 \text{ or } -1, \\ r_1 + r_2 + \cdots + r_i & \text{if } i \geq 1, \end{cases}$$

and

$$\Lambda_k^{(c)} \left[\begin{array}{c} \{x_i, y_i\} \\ [u, v] \end{array} \right]_q := \prod_{i=u}^{v} \frac{(x_i, y_i; q)_k}{(c/x_i, c/y_i; q)_k} \left(\frac{c}{x_i y_i} \right)^k .$$

$$\sum_{m=0}^{\infty} \frac{(a/x, y/x; q)_m}{(q, aq/y; q)_m} \left(\frac{x^2 q}{y} \right)^m \sum_{n=0}^{m} \frac{(q^{-m}, q^m a/x, y; q)_n}{(y/x; q)_n} q^n A_n$$
$$= \frac{(axq, xq; q)_\infty}{(aq, x^2 q; q)_\infty} \sum_{m=0}^{\infty} \frac{(1 - aq^{2m})(a, y; q)_m (a/x; q)_{2m}}{(1 - a)(q, aq/y; q)_m (axq; q)_{2m}} \left(\frac{x^2 q}{y} \right)^m \sum_{n=0}^{m} (q^{-m}, aq^m; q)_n q^n A_n .$$

Take

$$A_n = \sum_{k=0}^{[n/2]} \frac{(q, aq/\lambda; q^2)_k \lambda^k q^{k(2k-1)}}{(aq^{2k}, q; q)_{2k} (aq^{4k+1}, q; q)_{n-2k} (aq, q\lambda; q^2)_k} \Lambda_k^{(aq^2)} \left[\begin{array}{c} \{c_i, d_i\} \\ [1, t] \end{array} \right]_{q^2} .$$

Transformation E

Theorem

$$\sum_{m=0}^{\infty}\frac{(a/x,y/x;q)_m}{(q,aq/y;q)_m}\left(\frac{x^2q}{y}\right)^m\sum_{r_1,r_2\dots,r_t\geq 0}\prod_{j=1}^{t}\frac{(\frac{aq^2}{c_jd_j};q^2)_{r_j}(c_j,d_j;q^2)_{M_{j-1}}}{(q^2;q^2)_{r_j}(\frac{aq^2}{c_j},\frac{aq^2}{d_j};q^2)_{M_j}}\frac{(a^jq^{2j})^{r_{t-j+1}}}{(c_jd_j)^{M_{j-1}}}$$
$$\times\frac{(aq/\lambda;q^2)_{M_t}(q^{-m},aq^m/x,y;q)_{2M_t}(-\lambda)^{M_t}q^{M_t^2}}{(aq;q^2)_{2M_t}(\lambda q;q^2)_{M_t}(y/x;q)_{2M_t}a^{M_t}}$$
$$\times\,{}_5\phi_4\left[\begin{matrix}q^{-m+2M_t},aq^{m+2M_t}/x,yq^{2M_t},\sqrt{\lambda}q^{M_t},-\sqrt{\lambda}q^{M_t}\\ a^{1/2}q^{1/2+2M_t},-a^{1/2}q^{1/2+2M_t},\lambda q^{2M_t},yq^{2M_t}/x\end{matrix}\;\middle|\;q;q\right]$$
$$=\frac{(axq,xq;q)_\infty}{(aq,x^2q;q)_\infty}\sum_{m=0}^{\infty}\frac{(1-aq^{4m})(a,aq/\lambda;q^2)_m(y;q)_{2m}(a/x;q)_{4m}}{(1-a)(q^2,\lambda q;q^2)_m(aq/y;q)_{2m}(axq;q)_{4m}}$$
$$\times\Lambda_m^{(aq^2)}\left[\begin{matrix}\{c_i,d_i\}\\ [1,t]\end{matrix}\right]_{q^2}\left(\frac{\lambda x^4q^2}{y^2}\right)^m. \tag{27}$$

Remark. Taking $t=0$ gives Transformation D (22).

Transformation E

Let $x \to 0$, $y \to \infty$, $\lambda = -a^{1/2}$ and $c_1, d_1, \ldots, c_t, d_t \to \infty$ in (27). Then

$$\sum_{i,j\geq 0} \sum_{r_1,r_2,\ldots,r_t\geq 0} \frac{(-a^{1/2}q^{2M_t};q^2)_i a^{2i+j+M_1+\cdots+M_{t-1}+\frac{9}{2}M_t}}{(q;q)_i(q;q)_j(aq;q^2)_{2M_t+i}(-a^{1/2}q^{2M_t};q)_i}$$
$$\times \frac{q^{2i^2+j^2+2ij+(8i+4j)M_t+2(M_1^2+\cdots+M_{t-1}^2)+9M_t^2}}{(q^2;q^2)_{r_1}(q^2;q^2)_{r_2}\cdots(q^2;q^2)_{r_t}}$$
$$= \frac{1}{(aq;q)_\infty} \sum_{m=0}^{\infty} \frac{(1-aq^{4m})(a;q^2)_m}{(1-a)(q^2;q^2)_m}(-1)^m a^{(\frac{9}{2}+t)m} q^{2(t+5)m^2-m}.$$

Take $a = 1$ and $a = q^2$ and apply JTP.

Transformation E

For $t \geq 1$, we have

$$\sum_{i,j\geq 0}\sum_{r_1,r_2,\ldots,r_t\geq 0}\frac{(-q^{2M_t};q^2)_i}{(q;q)_i(q;q)_j(q;q^2)_{2M_t+i}(-q^{2M_t};q)_i}\\ \times\frac{q^{2i^2+j^2+2ij+(8i+4j)M_t+2(M_1^2+\cdots+M_{t-1}^2)+9M_t^2}}{(q^2;q^2)_{r_1}(q^2;q^2)_{r_2}\cdots(q^2;q^2)_{r_t}}\\ =\frac{(q^{2t+9},q^{2t+11},q^{4t+20};q^{4t+20})_\infty}{(q;q)_\infty}, \tag{28}$$

$$\sum_{i,j\geq 0}\sum_{r_1,r_2,\ldots,r_t\geq 0}\frac{(-q^{2M_t+1};q^2)_i}{(q;q)_i(q;q)_j(q;q^2)_{2M_t+i+1}(-q^{2M_t+1};q)_i}\\ \times\frac{q^{2i^2+j^2+2ij+4i+2j+(8i+4j+9)M_t+2(M_1^2+\cdots+M_{t-1}^2+M_1+\cdots+M_{t-1})+9M_t^2}}{(q^2;q^2)_{r_1}(q^2;q^2)_{r_2}\cdots(q^2;q^2)_{r_t}}\\ =\frac{(q^2,q^{4t+18},q^{4t+20};q^{4t+20})_\infty}{(q;q)_\infty}. \tag{29}$$

Transformation E

Let $x \to 0$, $y = (aq)^{1/2}$, $\lambda = -a^{1/2}$ and $c_1, d_1, \ldots, c_t, d_t \to \infty$ in (27). Then

$$\sum_{i,j\geq 0} \sum_{r_1,r_2,\ldots,r_t\geq 0} \frac{(-a^{1/2}q^{2M_t};q^2)_i(-1)^i a^{\frac{3}{2}i+j+M_1+\cdots+M_{t-1}+\frac{7}{2}M_t}}{(q;q)_i(q;q)_j(aq;q^2)_{2M_t+i}(-a^{1/2}q^{2M_t};q)_i(a^{1/2}q^{i+2M_t+1/2};q)_j}$$
$$\times \frac{q^{\frac{3}{2}i^2+j^2+2ij+(6i+4j)M_t+2(M_1^2+\cdots+M_{t-1}^2)+7M_t^2}}{(q^2;q^2)_{r_1}(q^2;q^2)_{r_2}\cdots(q^2;q^2)_{r_t}}$$
$$= \frac{1}{(aq;q)_\infty} \sum_{m=0}^{\infty} \frac{(1-aq^{4m})(a;q^2)_m}{(1-a)(q^2;q^2)_m}(-1)^m a^{(\frac{7}{2}+t)m} q^{2(t+4)m^2-m}.$$

Take $a = 1$ and $a = q^2$ and apply JTP.

Transformation E

For $t \geq 1$, we have

$$\sum_{i,j\geq 0}\sum_{r_1,r_2,\ldots,r_t\geq 0}\frac{(-q^{2M_t};q^2)_i(-1)^i}{(q;q)_i(q;q)_j(q;q^2)_{2M_t+i}(-q^{2M_t};q)_i(q^{i+2M_t+1/2};q)_j}$$
$$\times\frac{q^{\frac{3}{2}i^2+j^2+2ij+(6i+4j)M_t+2(M_1^2+\cdots+M_{t-1}^2)+7M_t^2}}{(q^2;q^2)_{r_1}(q^2;q^2)_{r_2}\cdots(q^2;q^2)_{r_t}}$$
$$=\frac{(q^{2t+7},q^{2t+9},q^{4t+16};q^{4t+16})_\infty}{(q;q)_\infty}, \tag{30}$$

$$\sum_{i,j\geq 0}\sum_{r_1,r_2,\ldots,r_t\geq 0}\frac{(-q^{2M_t+1};q^2)_i(-1)^i}{(q;q)_i(q;q)_j(q;q^2)_{2M_t+i+1}(-q^{2M_t+1};q)_i(q^{i+2M_t+3/2};q)_{i+j}}$$
$$\times\frac{q^{\frac{3}{2}i^2+j^2+2ij+3i+2j+(6i+4j+7)M_t+2(M_1^2+\cdots+M_{t-1}^2+M_1+\cdots+M_{t-1})+7M_t^2}}{(q^2;q^2)_{r_1}(q^2;q^2)_{r_2}\cdots(q^2;q^2)_{r_t}}$$
$$=\frac{(q^2,q^{4t+14},q^{4t+16};q^{4t+16})_\infty}{(q;q)_\infty}. \tag{31}$$

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Thank You!