Linked partition ideals and Schur's 1926 partition theorem

Shane Chern

Dalhousie University

chenxiaohang92@gmail.com

Specialty Seminar in Partitions and q-Series

@ Michigan Tech

Mar 10, 2022

Bonus: Schmidt-type Identities

Theorem (Andrews–Paule [2022, JNT])

Let $g(n)$ denote the number of partitions μ into parts $\mu_1 \geq \mu_2 \geq \mu_3 \geq \cdots$ such that $\mu_1 + \mu_3 + \mu_5 + \cdots = n$. Then $g(n) = p_2(n)$, the number of partitions of n into two colors.

- Let $\Gamma_i(\lambda)$ be the hook length of the i-th diagonal entry of the Durfee square of λ, ordering from top-left to bottom-right.

Theorem (C.–Yee [2021, submitted])

Let $\gamma_e(n)$ (resp. $\gamma_o(n)$) be the number of partitions λ such that its length $\ell(\lambda)$ and the length $D(\lambda)$ of its Durfee square has the same parity (resp. different parities) and such that the diagonal hook lengths satisfy $(\Gamma_1(\lambda)+1)+(\Gamma_3(\lambda)+1)+(\Gamma_5(\lambda)+1)+\cdots = n$. Then, $\gamma_e(n)-\gamma_o(n)$ equals the number of partitions of n into even parts. In particular, $\gamma_e(2n+1) = \gamma_o(2n+1)$.

- Analytic Proof: MacMahon's Partition Analysis
- Combinatorial Proof: A new involution on the set of partitions

Introduction

The theory of partitions was given to birth in a letter from Leibniz to Bernoulli in September 1674, in which Leibniz asked for the number of representations of a positive integer n as a sum of positive integers, which is now called the number of integer partitions of n, usually denoted by $p(n)$, if the order of the summands is not taken into account.

Definition. An *integer partition* of a positive integer n is a non-increasing sequence of positive integers whose sum equals n. These summands are called *parts* of this partition. We usually use $p(n)$ to denote the number of partitions of n. Conventionally, we also put $p(0) = 1$, which means that 0 has an *empty partition* $\emptyset$ containing no parts.

Introduction

Generating function of $p(n)$.

$$\sum_{n\geq 0} p(n)q^n = (1+q^{1\cdot 1}+q^{2\cdot 1}+q^{3\cdot 1}+\cdots)$$
$$\times(1+q^{1\cdot 2}+q^{2\cdot 2}+q^{3\cdot 2}+\cdots)$$
$$\times(1+q^{1\cdot 3}+q^{2\cdot 3}+q^{3\cdot 3}+\cdots)$$
$$\times\cdots$$
$$=\prod_{k\geq 1}(1+q^k+q^{2k}+q^{3k}+\cdots)$$
$$=\prod_{k\geq 1}\frac{1}{1-q^k}=\frac{1}{(q;q)_\infty}.$$

Definition. (q-Pochhammer symbols). Let $q\in\mathbb{C}$ be such that $|q|<1$. For $n\in\mathbb{N}\cup\{\infty\}$,

$$(A;q)_n := \prod_{k=0}^{n-1}(1-Aq^k).$$

Introduction

Congruence *conditions on the parts.*

- Let $RR_C(n)$ denote the number of partitions of n such that all parts are congruent to ± 1 modulo 5.

$$\begin{aligned}\sum_{n\geq 0} RR_C(n)q^n = {} & (1+q^{1\cdot 1}+q^{2\cdot 1}+q^{3\cdot 1}+\cdots) \\ & \times (1+q^{1\cdot 4}+q^{2\cdot 4}+q^{3\cdot 4}+\cdots) \\ & \times (1+q^{1\cdot 6}+q^{2\cdot 6}+q^{3\cdot 6}+\cdots) \\ & \times (1+q^{1\cdot 9}+q^{2\cdot 9}+q^{3\cdot 9}+\cdots) \\ & \times \cdots \\ = {} & \prod_{k\geq 0} \frac{1}{1-q^{5k+1}} \prod_{k\geq 0} \frac{1}{1-q^{5k+4}} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}.\end{aligned}$$

Introduction

Difference *conditions on the parts.*

- Let $RR_D(n)$ denote the number of partitions of n such that each two consecutive parts have difference at least 2.
- Notice that $x_{i+1} - x_i \geq 2 \implies x_{i+1} - (2i+1) \geq x_i - (2i-1)$

$$\begin{array}{ccccccccc} 1 & \leq & x_1 & < & x_2 & < & \cdots & < & x_k \\ & & | & & | & & & & | \\ & & 1 & & 3 & & & & 2k-1 \\ \hline 0 & \leq & x_1' & \leq & x_2' & \leq & \cdots & \leq & x_k' \end{array}$$

$$\sum_{n\geq 0} RR_D(n) q^n = \sum_{k\geq 0} \frac{q^{k^2}}{(q;q)_k}.$$

Introduction

Theorem (First Rogers–Ramanujan identity)

The number of partitions of a non-negative integer n into parts congruent to ± 1 modulo 5 is the same as the number of partitions of n such that each two consecutive parts have difference at least 2.

In other words, for $n \geq 0$,

$$RR_C(n) = RR_D(n).$$

Theorem (First Rogers–Ramanujan identity (analytic form))

We have

$$\frac{1}{(q, q^4; q^5)_\infty} = \sum_{n \geq 0} \frac{q^{n^2}}{(q;q)_n}.$$

Introduction

Identities of Rogers–Ramanujan type (*Partition-theoretic*):

- **Congruence conditions** $\Leftrightarrow$ **Difference conditions**
- Schur (1926)
- Gleißberg (1928): Extension of Schur
- Andrews–Gordon (1961, 1969): n-folder generalization of Rogers–Ramanujan
- Göllnitz–Gordon (1961, 1965)
- Recent conjectures of Kanade and Russell (2015)

Identities of Rogers–Ramanujan type (*q-theoretic*):

- q-**Product** $\Leftrightarrow$ q-**Sum**
- Rogers Mod 14 (1894, 1917)
- Bailey Mod 9 (1947)
- Slater's list (1952)

Schur

Theorem (Schur [1926])

Let $A(n)$ denote the number of partitions of n into parts congruent to ± 1 modulo 6. Let $B(n)$ denote the number of partitions of n into distinct nonmultiples of 3. Let $D(n)$ denote the number of partitions of n of the form $\mu_1 + \mu_2 + \cdots + \mu_s$ where $\mu_i - \mu_{i+1} \geq 3$ with strict inequality if $3 \mid \mu_i$. Then

$$A(n) = B(n) = D(n).$$

$A(n) = B(n)$:

- $$\sum_{n\geq 0} A(n)q^n = \frac{1}{(q;q^6)_\infty (q^5;q^6)_\infty}.$$
- $$\begin{aligned}\sum_{n\geq 0} B(n)q^n &= (1+q)(1+q^2)(1+q^4)(1+q^5)\cdots \\ &= (-q;q^3)_\infty(-q^2;q^3)_\infty.\end{aligned}$$

$A(n) = B(n)$:

- $$\begin{aligned}(-q;q^3)_\infty(-q^2;q^3)_\infty &= (1+q)(1+q^2)(1+q^4)(1+q^5)\cdots \\ &= \frac{(1-q^2)(1-q^4)(1-q^8)(1-q^{10})\cdots}{(1-q)(1-q^2)(1-q^4)(1-q^5)\cdots} \\ &= \frac{(q^2;q^6)_\infty(q^4;q^6)_\infty}{(q;q^3)_\infty(q^2;q^3)_\infty} \\ &= \frac{(q^2;q^6)_\infty(q^4;q^6)_\infty}{(q;q^6)_\infty(q^2;q^6)_\infty(q^4;q^6)_\infty(q^5;q^6)_\infty} \\ &= \frac{1}{(q;q^6)_\infty(q^5;q^6)_\infty}.\end{aligned}$$

Schur

$D(n)$: $\mu_1 + \mu_2 + \cdots + \mu_s$

- $\mu_i - \mu_{i+1} \geq 3$;
- $\mu_i - \mu_{i+1} > 3$ if $3 \mid \mu_i$.

Example. We decompose each partition in $\mathscr{D}$ into blocks $B_0, B_1, \ldots$ such that all parts between $3i+1$ and $3i+3$ fall into block B_i.

$$4 + 7 + 12 + 17 + 20 + 24$$
$$\Downarrow$$
$$() + (4) + (7) + (12) + () + (17) + (20) + (24)$$
$$\Downarrow$$
$$\phi^0(\emptyset) + \phi^3(1) + \phi^6(1) + \phi^9(3) + \phi^{12}(\emptyset) + \phi^{15}(2) + \phi^{18}(2) + \phi^{21}(3)$$
$$\Downarrow$$
$$\emptyset \to 1 \to 1 \to 3 \to \emptyset \to 2 \to 2 \to 3$$

We define operators ϕ^ℓ with $\ell \geq 0$ for partitions by adding ℓ to each part of the partition. In particular, $\phi^\ell(\emptyset) = \emptyset$ for all $\ell \geq 0$.

$D(n)$: $\mu_1 + \mu_2 + \cdots + \mu_s$

- $\mu_i - \mu_{i+1} \geq 3$;
- $\mu_i - \mu_{i+1} > 3$ if $3 \mid \mu_i$.

From the decomposition:

- Finite set of partitions $\Pi = \{\pi_1 = \emptyset, \pi_2 = (1), \pi_3 = (2), \pi_4 = (3)\}$.
- Further requirements:
  - $\pi_1 \to \{\pi_1, \pi_2, \pi_3, \pi_4\}$. *If* $\phi^{-3i}(B_i)$ *is* $\pi_1 = \emptyset$, *then* $\phi^{-3(i+1)}(B_{i+1})$ *can be any among* $\{\pi_1, \pi_2, \pi_3, \pi_4\}$.
  - $\pi_2 \to \{\pi_1, \pi_2, \pi_3, \pi_4\}$.
  - $\pi_3 \to \{\pi_1, \pi_3, \pi_4\}$. $\boxed{(3i+2) \to (3(i+1)+1)}$ ✗
  - $\pi_4 \to \{\pi_1\}$. $\boxed{(3i+3) \to (3(i+1)+1) \textit{ or } (3(i+1)+2) \textit{ or } (3(i+1)+3)}$ ✗

Span one linked partition ideals

Assume that we are given

- a finite set $\Pi = \{\pi_1, \pi_2, \ldots, \pi_K\}$ of integer partitions with $\pi_1 = \emptyset$, the empty partition,
- a *map of linking sets*, $\mathcal{L} : \Pi \to P(\Pi)$, the power set of Π, with especially, $\mathcal{L}(\pi_1) = \mathcal{L}(\emptyset) = \Pi$ and $\pi_1 = \emptyset \in \mathcal{L}(\pi_k)$ for any $1 \leq k \leq K$,
- and a positive integer T, called the *modulus*, which is greater than or equal to the largest part among all partitions in Π.

Span one linked partition ideals

Consider

- an infinite chain of partitions in Π:

$$\lambda_0 \to \lambda_1 \to \cdots \to \lambda_N \to \pi_1 \to \pi_1 \to \cdots$$

  ending with a series of empty partitions, such that $\lambda_i \in \mathcal{L}(\lambda_{i-1})$ for each i;
- an integer partition λ by

$$\lambda = \phi^0(\lambda_0) \oplus \phi^T(\lambda_1) \oplus \phi^{2T}(\lambda_2) \oplus \cdots \oplus \phi^{NT}(\lambda_N),$$

  where $\mu \oplus \nu$ is the partition constructed by collecting all parts in partitions μ and ν, and $\phi^m(\mu)$ is the partition obtained by adding m to each part of μ.

We collect all such partitions λ constructed as above and call this partition set a *span one linked partition ideal*, denoted by $\mathscr{I} = \mathscr{I}(\langle \Pi, \mathcal{L} \rangle, T)$.

Span one linked partition ideals

Define for any partition λ,

- $|\lambda|$: its size (aka. sum of all parts);
- $\sharp(\lambda)$: its length (aka. the number of parts);
- $s(\lambda)$: a statistic of $\lambda \in \mathscr{I}$ such that

$$s(\lambda) = s(\phi^T(\lambda))$$

and

$$s(\lambda) = s(\lambda_0) + s(\lambda_1) + \cdots + s(\lambda_N).$$

For each $1 \leq k \leq K$, we write

$$G_k(x) := \sum_{\substack{\lambda \in \mathscr{I} \\ \lambda_0 = \pi_k}} x^{\sharp(\lambda)} y^{s(\lambda)} q^{|\lambda|}.$$

Span one linked partition ideals

Then

$$\begin{pmatrix} G_1(x) \\ G_2(x) \\ \vdots \\ G_K(x) \end{pmatrix} = \mathscr{W}(x).\mathscr{A}.\begin{pmatrix} G_1(xq^T) \\ G_2(xq^T) \\ \vdots \\ G_K(xq^T) \end{pmatrix}$$

where the diagonal matrix $\mathscr{W}(x)$ is given by

$$\mathrm{diag}(x^{\sharp(\pi_1)}y^{s(\pi_1)}q^{|\pi_1|}, \ldots, x^{\sharp(\pi_K)}y^{s(\pi_K)}q^{|\pi_K|})$$

and the zero-one matrix $\mathscr{A}$ is given by

$$\mathscr{A}_{i,j} = \begin{cases} 1 & \text{if } \pi_j \in \mathcal{L}(\pi_i), \\ 0 & \text{if } \pi_j \notin \mathcal{L}(\pi_i). \end{cases}$$

Span one linked partition ideals

Let us write

$$\begin{pmatrix} F_1(x) \\ F_2(x) \\ \vdots \\ F_K(x) \end{pmatrix} = \mathscr{A} . \begin{pmatrix} G_1(x) \\ G_2(x) \\ \vdots \\ G_K(x) \end{pmatrix} .$$

Then

$$\begin{pmatrix} F_1(x) \\ F_2(x) \\ \vdots \\ F_K(x) \end{pmatrix} = \mathscr{A} . \mathscr{W}(x) . \begin{pmatrix} F_1(xq^T) \\ F_2(xq^T) \\ \vdots \\ F_K(xq^T) \end{pmatrix} .$$

Notice that we have $G_k(0) = 1$ if $k = 1$, and 0 otherwise. This is because the only non-vanishing term in $G_k(0)$ comes from the empty partition, which is exclusively counted by $G_1(x)$. Also, all entries in the first column of $\mathscr{A}$ are 1 since $\pi_1 \in \mathcal{L}(\pi_k)$ for all k. Hence, $F_1(0) = \cdots = F_K(0) = 1$.

Alladi–Gordon

Theorem (Gleißberg [1928])

Let $B(m,n)$ denote the number of partitions of n into m distinct nonmultiples of 3. Let $D(m,n)$ denote the number of partitions of n enumerated by $D(n)$ such that the total number of parts plus the number of multiples of 3 among the parts equals m. Then

$$B(m,n) = D(m,n).$$

- $\sharp_{a,M}(\lambda)$: the number of parts in λ that are congruent to a modulo M.

Theorem (Alladi–Gordon [1995]; C. [2022, Rocky Mountain J.])

$$\sum_{\lambda\in\mathscr{D}} x^{\sharp(\lambda)} y^{\sharp_{0,3}(\lambda)} q^{|\lambda|} = \sum_{n_1,n_2,n_3\ge 0} \frac{x^{n_1+n_2+n_3} y^{n_3}}{(q^3;q^3)_{n_1}(q^3;q^3)_{n_2}(q^3;q^3)_{n_3}} \times q^{3\binom{n_1}{2}+3\binom{n_2}{2}+6\binom{n_3}{2}+3n_1n_2+3n_2n_3+3n_3n_1+n_1+2n_2+3n_3}.$$

Remark. The proof of Alladi and Gordon relies on the technique of weighted words.

Alladi–Gordon

- $\mathscr{D}$ is a span one linked partition ideal $\mathscr{I}(\langle \Pi, \mathcal{L} \rangle, 3)$ with $\Pi = \{\pi_1 = \emptyset, \pi_2 = (1), \pi_3 = (2), \pi_4 = (3)\}$ and

$$\begin{cases} \mathcal{L}(\pi_1) = \{\pi_1, \pi_2, \pi_3, \pi_4\}, \\ \mathcal{L}(\pi_2) = \{\pi_1, \pi_2, \pi_3, \pi_4\}, \\ \mathcal{L}(\pi_3) = \{\pi_1, \pi_3, \pi_4\}, \\ \mathcal{L}(\pi_4) = \{\pi_1\}. \end{cases}$$

- Define for $1 \le k \le 4$,

$$G_k(x) := \sum_{\substack{\lambda \in \mathscr{D} \\ \lambda_0 = \pi_k}} x^{\sharp(\lambda)} y^{\sharp_{0,3}(\lambda)} q^{|\lambda|}.$$

Alladi–Gordon

- $$\begin{pmatrix} G_1(x) \\ G_2(x) \\ G_3(x) \\ G_4(x) \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & xq & & \\ & & xq^2 & \\ & & & xyq^3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} G_1(xq^3) \\ G_2(xq^3) \\ G_3(xq^3) \\ G_4(xq^3) \end{pmatrix}.$$

- $$\begin{pmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \\ F_4(x) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} G_1(x) \\ G_2(x) \\ G_3(x) \\ G_4(x) \end{pmatrix}.$$

Alladi–Gordon

- $$\begin{pmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \\ F_4(x) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & & & \\ & xq & & \\ & & xq^2 & \\ & & & xyq^3 \end{pmatrix} \cdot \begin{pmatrix} F_1(xq^3) \\ F_2(xq^3) \\ F_3(xq^3) \\ F_4(xq^3) \end{pmatrix}.$$

- $$\sum_{\lambda \in \mathscr{D}} x^{\sharp(\lambda)} y^{\sharp_{0,3}(\lambda)} q^{|\lambda|} = G_1(x) + G_2(x) + G_3(x) + G_4(x) = F_1(x).$$

- $$F_1(x) \stackrel{?}{=} \sum_{n_1,n_2,n_3 \geq 0} \frac{x^{n_1+n_2+n_3} y^{n_3}}{(q^3;q^3)_{n_1}(q^3;q^3)_{n_2}(q^3;q^3)_{n_3}} \times q^{3\binom{n_1}{2}+3\binom{n_2}{2}+6\binom{n_3}{2}+3n_1n_2+3n_2n_3+3n_3n_1+n_1+2n_2+3n_3}.$$

Alladi–Gordon

- Let R be a fixed positive integer. We fix a symmetric matrix $\underline{\boldsymbol{\alpha}} = (\alpha_{i,j}) \in \mathrm{Mat}_{R\times R}(\mathbb{N})$ and a vector $\underline{\mathbf{A}} = (A_r) \in \mathbb{N}_{>0}^R$. We also fix J vectors $\underline{\boldsymbol{\gamma_j}} = (\gamma_{j,r}) \in \mathbb{N}_{\geq 0}^R$ for $j = 1, 2, \ldots, J$. Let $x_1, x_2, \ldots, x_J$ and q be indeterminates such that the following q-multi-summation $H(\underline{\boldsymbol{\beta}}) = H(\beta_1, \ldots, \beta_R)$ converges.

$$H(\underline{\boldsymbol{\beta}}) := \sum_{n_1,\ldots,n_R\geq 0} \frac{x_1^{\sum_{r=1}^R \gamma_{1,r} n_r} \cdots x_J^{\sum_{r=1}^R \gamma_{J,r} n_r}}{(q^{A_1};q^{A_1})_{n_1} \cdots (q^{A_R};q^{A_R})_{n_R}} \times q^{\sum_{r=1}^R \alpha_{r,r}\binom{n_r}{2} + \sum_{1\leq i<j\leq R} \alpha_{i,j} n_i n_j + \sum_{r=1}^R \beta_r n_r}.$$

Alladi–Gordon

Lemma (C. [2022])

For $1 \leq r \leq R$*, we have*

$$H(\beta_1, \ldots, \beta_r, \ldots, \beta_R) = H(\beta_1, \ldots, \beta_r + A_r, \ldots, \beta_R) + x_1^{\gamma_{1,r}} \cdots x_J^{\gamma_{J,r}} q^{\beta_r} H(\beta_1 + \alpha_{r,1}, \ldots, \beta_r + \alpha_{r,r}, \ldots, \beta_R + \alpha_{r,R}).$$

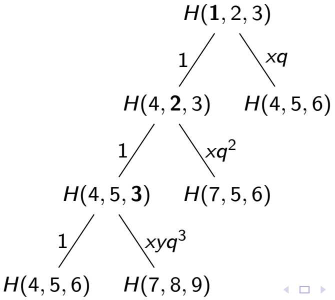
$$H(\beta_1, \ldots, \boldsymbol{\beta_r}, \ldots, \beta_R)$$

1 / $x_1^{\gamma_{1,r}} \cdots x_J^{\gamma_{J,r}} q^{\beta_r}$

$$H(\beta_1, \ldots, \beta_r + A_r, \ldots, \beta_R) \qquad H(\beta_1 + \alpha_{r,1}, \ldots, \beta_r + \alpha_{r,r}, \ldots, \beta_R + \alpha_{r,R})$$

Alladi–Gordon

- Choose $\underline{\alpha} = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 6 \end{pmatrix}$, $\underline{\gamma_1} = (1,1,1)$, $\underline{\gamma_2} = (0,0,1)$ and $\underline{\mathbf{A}} = (3,3,3)$ and write $x_1 = x$ and $x_2 = y$.
- $$\begin{pmatrix} H(1,2,3) \\ H(1,2,3) \\ H(4,2,3) \\ H(4,5,6) \end{pmatrix} \overset{?}{=} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & & & \\ & xq & & \\ & & xq^2 & \\ & & & xyq^3 \end{pmatrix} \cdot \begin{pmatrix} H(4,5,6) \\ H(4,5,6) \\ H(7,5,6) \\ H(7,8,9) \end{pmatrix}.$$



Alladi–Andrews

Theorem (Alladi [unpublished])

If we define $C(n)$ to be the number of partitions of n into odd parts with none appearing more than twice, then

$$C(n) = D(n).$$

Theorem (Andrews [2017])

Let $C(m,n)$ denote the number of partitions of n into m odd parts with none appearing more than twice. Let $D'(m,n)$ denote the number of partitions of n enumerated by $D(n)$ such that the total number of parts plus the number of even parts equals m. Then

$$C(m,n) = D'(m,n).$$

2 Proof of Theorem 5.2

We define $d_N(x,q) = d_N(x)$ to be the generating function for partitions of the type enumerated by $D(m,n)$ with the added condition that all parts be $\le N$.

We also define

$$\varepsilon(n) = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

Then for $n \ge 0$

$$d_{3n}(x) = d_{3n-1}(x) + x^{\varepsilon(3n)} q^{3n} d_{3n-4}(x), \tag{5.1}$$

$$d_{3n+1}(x) = d_{3n}(x) + x^{\varepsilon(3n+1)} q^{3n+1} d_{3n-2}(x), \tag{5.2}$$

$$d_{3n+2}(x) = d_{3n+1}(x) + x^{\varepsilon(3n+2)} q^{3n+2} d_{3n-1}(x), \tag{5.3}$$

with the initial condition $d_{-1}(x) = d_{-2}(x) = 1$, $d_{-4}(x) = 0$.

"From TOP to BOTTOM" vs "From BOTTOM to TOP"?

Theorem (Andrews–C.–Li [2022, JCTA])

$$\sum_{\lambda\in\mathscr{D}} x^{\sharp(\lambda)} y^{\sharp_{0,2}(\lambda)} q^{|\lambda|} = \sum_{n_1,n_2,n_3\geq 0} \frac{(-1)^{n_3} x^{n_1+n_2+2n_3} y^{n_2+n_3}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^6;q^6)_{n_3}} \times q^{4\binom{n_1}{2}+4\binom{n_2}{2}+18\binom{n_3}{2}+2n_1n_2+6n_2n_3+6n_3n_1+n_1+2n_2+9n_3}.$$

Remark. An *Andrews–Gordon type series* is of the form

$$\sum_{n_1,\ldots,n_r\geq 0} \frac{(-1)^{L_1(n_1,\ldots,n_r)} q^{Q(n_1,\ldots,n_r)+L_2(n_1,\ldots,n_r)}}{(q^{A_1};q^{A_1})_{n_1}\cdots(q^{A_r};q^{A_r})_{n_r}},$$

in which L_1 and L_2 are linear forms and Q is a quadratic form in $n_1,\ldots,n_r$.

Alladi–Andrews

- $\mathscr{D}$ is also the span one linked partition ideal $\mathscr{I}(\langle \Pi, \mathcal{L} \rangle, 6)$, where $\Pi = \{\pi_1 = \emptyset, \pi_2 = (1), \pi_3 = (2), \pi_4 = (3), \pi_5 = (4), \pi_6 = (1+4), \pi_7 = (5), \pi_8 = (1+5), \pi_9 = (2+5), \pi_{10} = (6), \pi_{11} = (1+6), \pi_{12} = (2+6)\}$ and

$$\begin{cases} \mathcal{L}(\pi_1) = \mathcal{L}(\pi_2) = \cdots = \mathcal{L}(\pi_6) = \{\pi_1, \pi_2, \ldots, \pi_{12}\}, \\ \mathcal{L}(\pi_7) = \mathcal{L}(\pi_8) = \mathcal{L}(\pi_9) = \{\pi_1, \pi_3, \pi_4, \pi_5, \pi_7, \pi_9, \pi_{10}, \pi_{12}\}, \\ \mathcal{L}(\pi_{10}) = \mathcal{L}(\pi_{11}) = \mathcal{L}(\pi_{12}) = \{\pi_1, \pi_5, \pi_7, \pi_{10}\}. \end{cases}$$

- Define for $1 \leq k \leq 12$,

$$G_k(x) := \sum_{\substack{\lambda \in \mathscr{D} \\ \lambda_0 = \pi_k}} x^{\sharp(\lambda)} y^{\sharp_{0,2}(\lambda)} q^{|\lambda|}.$$

Alladi–Andrews

- $$\begin{pmatrix} G_1(x) \\ G_2(x) \\ \vdots \\ G_{12}(x) \end{pmatrix} = \mathscr{W}(x).\mathscr{A}.\begin{pmatrix} G_1(xq^6) \\ G_2(xq^6) \\ \vdots \\ G_{12}(xq^6) \end{pmatrix}$$

  where

  $$\mathscr{W}(x) = \mathrm{diag}(1, xq, xyq^2, xq^3, xyq^4, x^2yq^5, xq^5, x^2q^6, x^2yq^7, xyq^6, x^2yq^7, x^2y^2q^8)$$

  and

  $$\mathscr{A} = \begin{pmatrix} 1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&0&1&1&1&0&1&0&1&1&0&1 \\ 1&0&1&1&1&0&1&0&1&1&0&1 \\ 1&0&1&1&1&0&1&0&1&1&0&1 \\ 1&0&0&0&1&0&1&0&0&1&0&0 \\ 1&0&0&0&1&0&1&0&0&1&0&0 \\ 1&0&0&0&1&0&1&0&0&1&0&0 \end{pmatrix}.$$

Alladi–Andrews

- $$\begin{pmatrix} F_1(x) \\ F_2(x) \\ \vdots \\ F_{12}(x) \end{pmatrix} = \mathscr{A}.\begin{pmatrix} G_1(x) \\ G_2(x) \\ \vdots \\ G_{12}(x) \end{pmatrix}.$$

- $$\begin{pmatrix} F_1(x) \\ F_2(x) \\ \vdots \\ F_{12}(x) \end{pmatrix} = \mathscr{A}.\mathscr{W}(x).\begin{pmatrix} F_1(xq^6) \\ F_2(xq^6) \\ \vdots \\ F_{12}(xq^6) \end{pmatrix}.$$

- $$\sum_{\lambda\in\mathscr{D}} x^{\sharp(\lambda)} y^{\sharp_{0,2}(\lambda)} q^{|\lambda|} = G_1(x) + G_2(x) + \cdots + G_{12}(x) = F_1(x).$$

Alladi–Andrews

- $$F_1(x) = F_2(x) = \cdots = F_6(x) =: A_1(x),$$
  $$F_7(x) = F_8(x) = F_9(x) =: A_2(x),$$
  $$F_{10}(x) = F_{11}(x) = F_{12}(x) =: A_3(x).$$

- $$\begin{pmatrix} A_1(x) \\ A_2(x) \\ A_3(x) \end{pmatrix} = \mathscr{M} \cdot \begin{pmatrix} A_1(xq^6) \\ A_2(xq^6) \\ A_3(xq^6) \end{pmatrix}$$

  where

  $$\mathscr{M} = \begin{pmatrix} 1 + xq + xyq^2 + xq^3 + xyq^4 + x^2yq^5 & xq^5 + x^2q^6 + x^2yq^7 & xyq^6 + x^2yq^7 + x^2y^2q^8 \\ 1 + xyq^2 + xq^3 + xyq^4 & xq^5 + x^2yq^7 & xyq^6 + x^2y^2q^8 \\ 1 + xyq^4 & xq^5 & xyq^6 \end{pmatrix}.$$

- Using an algorithm of C. and Li,

$$\begin{aligned}
0 &= \left[1 + x(q^7 + yq^8)\right] A_1(x)\\
&\quad - \left[1 + x(q + q^3 + q^5 + q^7 + yq^2 + yq^4 + yq^6 + yq^8)\right.\\
&\qquad + x^2(q^6 + q^8 + q^{10} + yq^5 + 2yq^7 + 2yq^9 + 2yq^{11} + yq^{13} + y^2q^8 + y^2q^{10} + y^2q^{12})\\
&\qquad \left. + x^3(yq^{12} + yq^{14} + y^2q^{13} + y^2q^{15})\right] A_1(xq^6)\\
&\quad + \left[x^2yq^{15} + x^3(-q^{21} + yq^{16} + y^2q^{17} - y^3q^{24})\right.\\
&\qquad + x^4(-q^{22} - yq^{23} + y^2q^{30} - y^3q^{25} - y^4q^{26})\\
&\qquad \left. + x^5(y^2q^{31} + y^3q^{32})\right] A_1(xq^{12}).
\end{aligned}$$

- Let $A_1(x) = \sum_{M\geq 0} a(M)x^M$. For any $M \geq 0$,

$$\begin{aligned}
0 &= q^{12M}(y^2q^{31} + y^3q^{32})a(M)\\
&\quad + q^{12(M+1)}(-q^{22} - yq^{23} + y^2q^{30} - y^3q^{25} - y^4q^{26})a(M+1)\\
&\quad + \left[-q^{6(M+2)}(yq^{12} + yq^{14} + y^2q^{13} + y^2q^{15})\right.\\
&\qquad \left. + q^{12(M+2)}(-q^{21} + yq^{16} + y^2q^{17} - y^3q^{24})\right] a(M+2)\\
&\quad + \left[-q^{6(M+3)}(q^6 + q^8 + q^{10} + yq^5 + 2yq^7 + 2yq^9 + 2yq^{11} + yq^{13} + y^2q^8 + y^2q^{10} + y^2q^{12})\right.\\
&\qquad \left. + q^{12(M+3)}yq^{15}\right] a(M+3)\\
&\quad + \left[(q^7 + yq^8) - q^{6(M+4)}(q + q^3 + q^5 + q^7 + yq^2 + yq^4 + yq^6 + yq^8)\right] a(M+4)\\
&\quad + \left[1 - q^{6(M+5)}\right] a(M+5).
\end{aligned}$$

- $$\begin{aligned} a(0) &= 1,\\ a(1) &= \frac{q(1+yq)}{1-q^2},\\ a(2) &= \frac{q^5(q-q^7+y+yq^2-yq^4-yq^{10}+y^2q^3-y^2q^9)}{(1-q^2)(1-q^4)(1-q^6)},\\ a(3) &= \frac{q^{12}(1+yq)(q^3+y+yq^2-yq^4+yq^8+y^2q^5)}{(1-q^2)(1-q^4)(1-q^6)}. \end{aligned}$$
- Assume the ansatz that $A_1(x)$ can be represented as an Andrews–Gordon series.
- From $a(1)$, it is natural to expect summations of the form:

$$\sum_{n_1\geq 0}\frac{q^{?}x^{n_1}}{(q^2;q^2)_{n_1}} \qquad \text{and} \qquad \sum_{n_2\geq 0}\frac{q^{?}x^{n_2}y^{n_2}}{(q^2;q^2)_{n_2}}.$$

- From $a(2)$, it is also highly possible that an extra summation is needed:

$$\sum_{n_3\geq 0}\frac{(-1)^{?}q^{?}x^{2n_3}y^{n_3}}{(q^6;q^6)_{n_3}}.$$

Alladi–Andrews

- $$A_1(x) \overset{?}{=} \sum_{n_1,n_2,n_3\geq 0} \frac{(-1)^{n_3} x^{n_1+n_2+2n_3} y^{n_2+n_3}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^6;q^6)_{n_3}} \times q^{4\binom{n_1}{2}+4\binom{n_2}{2}+18\binom{n_3}{2}+2n_1n_2+6n_2n_3+6n_3n_1+n_1+2n_2+9n_3}.$$

- $$\sum_{M\geq 0} \tilde{a}(M)x^M = \sum_{n_1,n_2,n_3\geq 0} \frac{(-1)^{n_3} x^{n_1+n_2+2n_3} y^{n_2+n_3}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^6;q^6)_{n_3}} \times q^{4\binom{n_1}{2}+4\binom{n_2}{2}+18\binom{n_3}{2}+2n_1n_2+6n_2n_3+6n_3n_1+n_1+2n_2+9n_3}.$$

- *Mathematica* packages of the Research Institute for Symbolic Computation (RISC) at Johannes Kepler University in Linz, Austria.

Alladi–Andrews

- `qMultiSum` implemented by Riese.

```
In[•]:= (********************
 Computing the recurrence for ã(M) using the qMultiSum package.
  ********************)

ClearAll[M, n1, n2, n3, U1, U2, U3, y];
U1 = 1;
U2 = 2;
U3 = 9;
n1 = M - n2 - 2 n3;
summand = ((-1)^n3 q^(4 Binomial[n1,2]+4 Binomial[n2,2]+18 Binomial[n3,2]+2 n1 n2+6 n2 n3+6 n3 n1+U1 n1+U2 n2+U3 n3) y^(n2+n3)) /
    (qPochhammer[q^2, q^2, n1] qPochhammer[q^2, q^2, n2] qPochhammer[q^6, q^6, n3]);
stru = qFindStructureSet[summand, {M}, {n2, n3}, {2}, {2, 2}, {2, 2}, qProtocol → True]
rec = qFindRecurrence[summand, {M}, {n2, n3}, {2}, {2, 2}, {2, 2}, qProtocol → True,
  StructSet → stru[[1]]]
sumrec = qSumRecurrence[rec]
```

Out[•]=

$$\{q^{24+12M} y^2 (1 + q^{22+6M} + 2qy + q^{23+6M} y + q^2 y^2 + q^{24+6M} y^2)\, \mathrm{SUM}[M] -$$
$$q^{27+12M} (1 + qy) (1 + q^{22+6M} + qy + q^2 y^2 - q^8 y^2 + q^{24+6M} y^2 + q^3 y^3 + q^4 y^4 + q^{26+6M} y^4)\, \mathrm{SUM}[1 + M] +$$
$$q^{17+6M} (q^{15+6M} - q^{21+6M} - y - q^2 y + 2q^{16+6M} y - 2q^{22+6M} y - q^{24+6M} y + q^{38+12M} y -$$
$$2qy^2 - 2q^3 y^2 + 3q^{17+6M} y^2 - 2q^{23+6M} y^2 - q^{25+6M} y^2 + q^{39+12M} y^2 - q^2 y^3 - q^4 y^3 +$$
$$2q^{18+6M} y^3 - 2q^{24+6M} y^3 - q^{26+6M} y^3 + q^{40+12M} y^3 + q^{19+6M} y^4 - q^{25+6M} y^4)\, \mathrm{SUM}[2 + M] -$$
$$q^{17+6M} (1 - q + q^2) (1 + q + q^2) (1 + qy) (1 + q^{20+6M} + qy + q^3 y + q^{21+6M} y + q^2 y^2 + q^{22+6M} y^2)$$
$$\mathrm{SUM}[3 + M] - (-1 + q^{4+M}) (1 + q^{4+M}) (1 - q^{4+M} + q^{8+2M}) (1 + q^{4+M} + q^{8+2M})$$
$$(1 + q^{16+6M} + 2qy + q^{17+6M} y + q^2 y^2 + q^{18+6M} y^2)\, \mathrm{SUM}[4 + M] == 0\}$$

Alladi–Andrews

- Let $d(M) := a(M) - \tilde{a}(M)$.
- `qGeneratingFunctions` implemented by Koutschan.

```
In[•]:= (********************
         Computing the recurrence for a(M) - ã(M) using the qGeneratingFunctions package.
         ********************)

        sumrec1 = {Rec == 0};
        sumrec2 = sumrec;
        ClearAll[M, y];
        QREPlus[sumrec1, sumrec2, SUM[M]]
```

Out[•]=

$$\{-q^{29}(-1+q^M)(1+q^M)(1-q^M+q^{2M})(1+q^M+q^{2M})\,\mathrm{SUM}[M] == -q^{12M}y^2(1+qy)\,\mathrm{SUM}[-5+M] - q^{3+12M}(-1-qy+q^8y^2-q^3y^3-q^4y^4)\,\mathrm{SUM}[-4+M] + q^{9+6M}(1+qy)(q^{5+6M}+q^{14}y+q^{16}y-q^{6M}y-q^{6+6M}y+q^{7+6M}y^2)\,\mathrm{SUM}[-3+M] + q^{20+6M}(q^3+q^5+q^7+q^2y+2q^4y+2q^6y+2q^8y+q^{10}y-q^{6M}y+q^5y^2+q^7y^2+q^9y^2)\,\mathrm{SUM}[-2+M] - q^{24}(q^{12}-q^{6M}-q^{2+6M}-q^{4+6M}-q^{6+6M})(1+qy)\,\mathrm{SUM}[-1+M]\}$$

- Verify $a(M) = \tilde{a}(M)$ for $0 \leq M \leq 4$.

Andrews

Theorem (Andrews [2000])

We consider partitions in which odd parts appear at most once, even parts appear at most twice, and the difference between two parts can never be 1 *and can be* 2 *only if both are odd. Let* $E(n)$ *denote the weighted count of these partitions with weight* $(-1)^\tau$ *for each partition that has exactly* τ *parts that appear twice. Then*

$$A(n) = B(n) = C(n) = D(n) = E(n).$$

- $\tau(\lambda)$: the number of different parts in λ that appear twice.

Theorem (C. [2021])

$$\sum_{\lambda \in \mathscr{E}} x^{\sharp(\lambda)} y^{\tau(\lambda)} q^{|\lambda| - \sharp(\lambda)(\sharp(\lambda)-1)} = \frac{(-xq^2; q^2)_\infty}{\prod_{n \geq 0} (1 - xq^{2n+1} - x^2 y q^{4n+2})}.$$

Andrews

- $\mathscr{E}$ is the span one linked partition ideal $\mathscr{I}(\langle \Pi, \mathcal{L} \rangle, 2)$, where $\Pi = \{\pi_1 = \emptyset, \pi_2 = (1), \pi_3 = (2), \pi_4 = (2+2)\}$ and

$$\begin{cases} \mathcal{L}(\pi_1) = \mathcal{L}(\pi_2) = \{\pi_1, \pi_2, \pi_3, \pi_4\}, \\ \mathcal{L}(\pi_3) = \mathcal{L}(\pi_4) = \{\pi_1\}. \end{cases}$$

- Define for $1 \leq k \leq 4$,

$$G_k(x) := \sum_{\substack{\lambda \in \mathscr{E} \\ \lambda_0 = \pi_k}} x^{\sharp(\lambda)} y^{\tau(\lambda)} q^{|\lambda|}.$$

Andrews

- $$\begin{pmatrix} G_1(x) \\ G_2(x) \\ G_3(x) \\ G_4(x) \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & xq & & \\ & & xq^2 & \\ & & & x^2yq^4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} G_1(xq^2) \\ G_2(xq^2) \\ G_3(xq^2) \\ G_4(xq^2) \end{pmatrix}.$$

- $$\begin{pmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \\ F_4(x) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} G_1(x) \\ G_2(x) \\ G_3(x) \\ G_4(x) \end{pmatrix}.$$

Andrews

- $$\begin{pmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \\ F_4(x) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & & & \\ & xq & & \\ & & xq^2 & \\ & & & x^2yq^4 \end{pmatrix} \cdot \begin{pmatrix} F_1(xq^2) \\ F_2(xq^2) \\ F_3(xq^2) \\ F_4(xq^2) \end{pmatrix}.$$

- $$\sum_{\lambda \in \mathscr{E}} x^{\sharp(\lambda)} y^{\tau(\lambda)} q^{|\lambda|} = G_1(x) + G_2(x) + G_3(x) + G_4(x) = F_1(x).$$

- $$F_1(x) = (1 + xq)F_1(xq^2) + (xq^2 + x^2yq^4)F_1(xq^4).$$

q-Borel operators:

Definition

Let $\mathbb{K}$ be a field. Let $F(x) = \sum_{n\geq 0} f(n)x^n \in \mathbb{K}(q)[[x]]$. We define the operator $\mathcal{B}_k$ for $k \in \mathbb{Z}$ by

$$\mathcal{B}_k\big(F(x)\big) := \sum_{n\geq 0} f(n)q^{-k\binom{n}{2}}x^n.$$

Lemma (C. [2021])

Let $F(x) \in \mathbb{K}(q)[[x]]$. For any integers k and N, and nonnegative integer M, we have

$$\mathcal{B}_k\big(x^M F(xq^N)\big) = x^M q^{-k\binom{M}{2}}\mathcal{B}_k\big(F(xq^{N-kM})\big).$$

Andrews

- Applying $\mathcal{B}_2$ and defining $Q(x) := \mathcal{B}_2\big(F_1(x)\big)$:

$$(1 - xq - x^2yq^2)Q(x) = (1 + xq^2)Q(xq^2).$$

-

$$Q(0) = P(0) = 1.$$

-

$$Q(x) = \prod_{n\geq 0} \frac{1 + xq^{2n+2}}{1 - xq^{2n+1} - x^2yq^{4n+2}}.$$

-

$$\begin{aligned} Q(x) &= \mathcal{B}_2\big(F_1(x)\big) \\ &= \mathcal{B}_2\left(\sum_{\lambda\in\mathscr{E}} x^{\sharp(\lambda)}y^{\tau(\lambda)}q^{|\lambda|}\right) \\ &= \sum_{\lambda\in\mathscr{E}} x^{\sharp(\lambda)}y^{\tau(\lambda)}q^{|\lambda|-2\binom{\sharp(\lambda)}{2}}. \end{aligned}$$

Andrews

Remark.

- *continuous q-Hermite polynomials* $H_n(x;q)$:

$$H_n(x;q) := e^{\mathrm{i}n\theta}\,{}_2\phi_0\left(\begin{matrix} q^{-n}, 0 \\ - \end{matrix}; q, q^n e^{-2\mathrm{i}\theta}\right) \qquad (\text{with } x = \cos\theta),$$

  where the basic hypergeometric series ${}_r\phi_s$ is defined by

$${}_r\phi_s\left(\begin{matrix} a_1, a_2 \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; q, z\right) := \sum_{n\geq 0} \frac{(a_1;q)_n \cdots (a_r;q)_n}{(q;q)_n (b_1;q)_n \cdots (b_s;q)_n} \left((-1)^n q^{\binom{n}{2}}\right)^{s-r+1} z^n.$$

- A family of q-orthogonal polynomials in the basic Askey scheme.

Corollary (C. [2021])

$$\sum_{\lambda\in\mathscr{E}} x^{\sharp(\lambda)} y^{\tau(\lambda)} q^{|\lambda|} = \sum_{M,N\geq 0} \frac{q^{2\binom{M}{2}+4\binom{N}{2}+2MN+M+2N} x^{M+N} t_M(y)}{(q^2;q^2)_M (q^2;q^2)_N},$$

where

$$t_M(y) = (-\mathrm{i})^M y^{M/2} H_M\left(\tfrac{\mathrm{i}}{2} y^{-1/2}; q^2\right).$$

Thank You!