

General coefficient-vanishing results associated with theta series

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Background

- **Motivation.** Given a Laurent series $G(q) = \sum_n g_n q^n$, can we find an arithmetic progression $Mn + w$ such that $g_{Mn+w} = 0$?
- **Ramanujan's theta series.**

$$\begin{aligned} f(a, b) &= \sum_{n \in \mathbb{Z}} a^{n(n+1)/2} b^{n(n-1)/2} \\ &= (-a, -b, ab; ab)_\infty. \end{aligned}$$

- We are interested in the case where $G(q)$ is a product/quotient of theta series.

Background

- **Mike Hirschhorn (2019).** If

$$\sum_n g(n)q^n = (-q, -q^4; q^5)_\infty (q, q^9; q^{10})_\infty^3,$$

then

$$g(5n + 2) = 0.$$

- **Jimmy Mc Laughlin (2021).** If for $b \in \{1, 2, 3\}$

$$\sum_n g(n)q^n = (q^b, q^{7-b}; q^7)_\infty (q^{7+2b}, q^{7-2b}; q^{14})_\infty^3,$$

then

$$g(7n + 4b) = 0.$$

Results

•

$$\sum_n \alpha_{i,j,r,\ell,m}(n)q^n := \frac{(q^i, q^{r-i}; q^r)_\infty^\ell}{(q^j, q^{r-j}; q^r)_\infty^m},$$

$$\sum_n \beta_{i,j,r,\ell,m}(n)q^n := \frac{(q^i, q^{r-i}; q^r)_\infty^\ell}{(-q^j, -q^{r-j}; q^r)_\infty^m},$$

$$\sum_n \gamma_{i,j,r,\ell,m}(n)q^n := (-q^i, -q^{r-i}; q^r)_\infty^\ell (q^j, q^{2r-j}; q^{2r})_\infty^m,$$

$$\sum_n \delta_{i,j,r,\ell,m}(n)q^n := (q^i, q^{r-i}; q^r)_\infty^\ell (-q^j, -q^{2r-j}; q^{2r})_\infty^m,$$

$$\sum_n \epsilon_{i,j,r,\ell,m}(n)q^n := (q^i, q^{r-i}; q^r)_\infty^\ell (q^j, q^{2r-j}; q^{2r})_\infty^m,$$

$$\sum_n \phi_{i,j,r,\ell,m}(n)q^n := (-q^i, -q^{r-i}; q^r)_\infty^\ell (q^j, q^{r-j}; q^r)_\infty^m,$$

$$\sum_n \psi_{i,j,r,\ell,m}(n)q^n := (q^i, q^{r-i}; q^r)_\infty^\ell (q^j, q^{r-j}; q^r)_\infty^m.$$

Results

Theorem (C. & Tang (2022))

$$\begin{aligned}\{\phi, \psi\}_{k, 2k, (2\ell+8m+5)\mu, 2\ell+1, 2m+1}((2\ell+8m+5)n + (2\ell+6m+4)k) &= 0, \\ \{\alpha, \beta\}_{2k, k, (8\ell+6m+3)\mu, 2\ell+2m+1, 2m+1}((8\ell+6m+3)n + (6\ell+4m+2)k) &= 0, \\ \{\alpha, \beta\}_{2k, k, (4\ell+6m+5)\mu, 2\ell+4m+3, 4m+2}((4\ell+6m+5)n + (2\ell+2m+2)k) &= 0, \\ \{\gamma, \delta\}_{k, (4\ell+2m+3)\mu+k, (4\ell+2m+3)\mu, 2\ell+1, 2m+1}((4\ell+2m+3)n + (3\ell+m+2)k) &= 0, \\ \{\gamma, \epsilon\}_{k, (2\ell+4m+3)\mu+2k, (2\ell+4m+3)\mu, 2\ell+1, 2m+1}((2\ell+4m+3)n + (2\ell+2m+2)k) &= 0, \\ \{\gamma, \delta\}_{k, k, (4\ell+2m+3)\mu, 2\ell+1, 2m+1}((4\ell+2m+3)n + (\ell+m+1)k) &= 0, \\ \{\gamma, \epsilon\}_{k, 4k, (2\ell+16m+9)\mu, 2\ell+1, 2m+1}((2\ell+16m+9)n + (2\ell+12m+7)k) &= 0, \\ \{\delta, \epsilon\}_{2k, k, (16\ell+2m+9)\mu, 2\ell+1, 2m+1}((16\ell+2m+9)n + (10\ell+2m+6)k) &= 0.\end{aligned}$$

Examples. (i). In the sixth equation, we choose $(k, \mu, \ell, m) = (1, 1, 0, 1)$, and get $\gamma_{1,1,5,1,3}(5n+2) = 0$ while $\sum_n \gamma_{1,1,5,1,3}(n)q^n = (-q, -q^4; q^5)_\infty(q, q^9; q^{10})_1^3$ is the series in Hirschhorn's example.

(ii). In the fifth equation, we choose $(k, \mu, \ell, m) = (b, 1, 0, 1)$, and get $\epsilon_{b,7+2b,7,1,3}(7n+4b) = 0$ while $\sum_n \epsilon_{b,7+2b,7,1,3}(n)q^n = (q^b, q^{7-b}; q^7)_\infty(q^{7+2b}, q^{7-2b}; q^{14})_1^3$ is the series in Mc Laughlin's example.

Results

Theorem (C. & Tang (2022))

If $\gcd(2\ell + 1, 2m + 1) = 1$,

$$\{\gamma, \delta\}_{(2m+1)k, (2\ell+1)k, (2\ell+4m+3)\mu, 2\ell+1, 2m+1}((2\ell + 4m + 3)n - 2(2m + 1)^2 k) = 0;$$

if $\gcd(2\ell + 1, 2m + 2) = 1$,

$$\{\delta, \epsilon\}_{(2m+2)k, (2\ell+1)k, (2\ell+4m+5)\mu, 2\ell+1, 2m+2}((2\ell + 4m + 5)n - 2(2m + 2)^2 k) = 0;$$

if $\gcd(2\ell + 2, 2m + 1) = 1$,

$$\{\gamma, \epsilon\}_{(2m+1)k, (4\ell+4)k, (4\ell+2m+5)\mu, 2\ell+2, 2m+1}((4\ell + 2m + 5)n - 3(2\ell + 2)^2 k) = 0.$$

Expanding and pairing powers of theta series

- **Ramanujan's Notebooks, Chapter XVI, Entry 29.** If $ab = cd$, then

$$f(a, b)f(c, d) + f(-a, -b)f(-c, -d) = 2f(ac, bd)f(ad, bc),$$

and

$$f(a, b)f(c, d) - f(-a, -b)f(-c, -d) = 2af(b/c, ac^2d)f(b/d, acd^2).$$

- Setting $a = c = q^k$ and $b = d = q^{-k+M}$, and then summing them,
- Define the huffing operator for $G(q) = \sum_n g_n q^n$ a Laurent series and M a positive integer by

$$H_M(G(q)) := \sum_n g_{Mn} q^{Mn}.$$

It a property that for any given series F in q^M ,

$$H_M(G(q) \cdot F(q^M)) = F(q^M) \cdot H_M(G(q)).$$

Expanding and pairing powers of theta series

- Schröter (1854).

$$\begin{aligned} & f(q^A x, q^A/x) f(q^B y, q^B/y) \\ &= \sum_{n=0}^{A+B-1} q^{An^2} x^n f(q^{A+B+2An} x/y, q^{A+B-2An} y/x) \\ &\quad \times f(q^{AB(A+B+2n)} (x^B y^A), q^{AB(A+B-2n)} / (x^B y^A)). \end{aligned}$$

In particular,

$$f(qz, q/z)^2 = \sum_{n=0}^1 q^{n^2} z^n f(q^{2+2n}, q^{2-2n}) f(q^{2+2n} z^2, q^{2-2n}/z^2).$$

Expanding and pairing powers of theta series

- **Mc Laughlin (2019).** For $m \geq 3$,

$$\begin{aligned} f(qz, q/z)^m &= \sum_{n_1=0}^1 \sum_{n_2=0}^2 \cdots \sum_{n_{m-1}=0}^{m-1} z^{n_{m-1}} q^{n_1^2 + (n_2 - n_1)^2 + \cdots + (n_{m-1} - n_{m-2})^2} \\ &\quad \times f(q^{2+2n_1}, q^{2-2n_1}) f(q^{m+2n_{m-1}} z^m, q^{m-2n_{m-1}} / z^m) \\ &\quad \times \prod_{i=2}^{m-1} f(q^{i(i+1)+2(i+1)n_{i-1}-2in_i}, q^{i(i+1)-2(i+1)n_{i-1}+2in_i}). \end{aligned}$$

Expanding and pairing powers of theta series

- Write the theta power

$$f((-1)^\kappa q^{k+A'}, (-1)^\kappa q^{-k+(A-A')})^m = \sum \mathcal{AF},$$

where

- \mathcal{A} is a theta series, usually times a power of -1 and a power of q ;
- \mathcal{F} is a series in q^A .

- **Quintuple product identity.**

$$\frac{f(-q^{2k}, -q^{\mu-2k}) f(-q^\mu, -q^{2\mu})}{f((-1)^\kappa q^k, (-1)^\kappa q^{\mu-k})} = f((-1)^\kappa q^{3k+\mu}, (-1)^\kappa q^{-3k+2\mu}) + (-1)^{\kappa+1} q^k f((-1)^\kappa q^{3k+2\mu}, (-1)^\kappa q^{-3k+\mu}).$$

Expanding and pairing powers of theta series

- **Hidden symmetry (This is NOT trivial!).** For each summand \mathcal{AF} with only one exception, there exists a companion summand $\mathcal{A}'\mathcal{F}'$ such that $\mathcal{F} = \mathcal{F}'$.



$$f((-1)^\kappa q^{k+A'}, (-1)^\kappa q^{-k+(A-A')})^m = \mathcal{A}_0 \mathcal{F}_0 + \sum (\mathcal{A}_I + \mathcal{A}_{II}) \mathcal{F}.$$



$$\left(\frac{f(-q^{2k}, -q^{\mu-2k})}{f((-1)^\kappa q^k, (-1)^\kappa q^{\mu-k})} \right)^m = \sum (\mathcal{A}_I + \mathcal{A}_{II}) \mathcal{F}.$$

The exceptional unpaired summand vanishes.

Cancelation under the action of the huffing operator

- Reformulate our seven series as

$$\left(\sum \mathcal{A} \mathcal{F} \right) \cdot \left(\sum \mathcal{B} \mathcal{G} \right),$$

or with our pairing process,

$$\left(\mathcal{A}_0 \mathcal{F}_0 + \sum (\mathcal{A}_I + \mathcal{A}_{II}) \mathcal{F} \right) \cdot \left(\mathcal{B}_0 \mathcal{G}_0 + \sum (\mathcal{B}_I + \mathcal{B}_{II}) \mathcal{G} \right),$$

where \mathcal{F} and \mathcal{G} (including \mathcal{F}_0 and \mathcal{G}_0) are series in q^M for a certain M .

Cancelation under the action of the huffing operator

- **An important observation from existing coefficient-vanishing results:**

We either encounter a series of cancelations according to the pairing process:

$$H_M(\mathcal{A}_0 \mathcal{B}_0) = 0,$$

$$H_M(\mathcal{A}_0 \mathcal{B}_I) = \pm H_M(\mathcal{A}_0 \mathcal{B}_{II}),$$

$$H_M(\mathcal{B}_0 \mathcal{A}_I) = \pm H_M(\mathcal{B}_0 \mathcal{A}_{II}),$$

$$H_M(\mathcal{A}_I \mathcal{B}_I) = \pm H_M(\mathcal{A}_{II} \mathcal{B}_{II}),$$

$$H_M(\mathcal{A}_I \mathcal{B}_{II}) = \pm H_M(\mathcal{A}_{II} \mathcal{B}_I),$$

or we uniformly have

$$H_M(\mathcal{A} \mathcal{B}) = 0.$$

Cancelation under the action of the huffing operator

- Consider

$$\begin{aligned}\mathcal{H}(q) := & q^w f\left((-1)^\kappa q^{u+A'M}, (-1)^\kappa q^{-u+(A-A')M}\right) \\ & \times f\left((-1)^\lambda q^{v+B'M}, (-1)^\lambda q^{-v+(B-B')M}\right).\end{aligned}$$

- Rewrite in the summation form

$$\mathcal{H}(q) = \sum_{m,n \in \mathbb{Z}} (-1)^{\kappa m + \lambda n} q^{AM\binom{m}{2} + A'Mm + BM\binom{n}{2} + B'Mn + um + vn + w}.$$

- Determine the solution set $\{(m, n) : m, n \in \mathbb{Z}\}$ of the linear congruence

$$um + vn + w \equiv 0 \pmod{M}.$$

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$$H_M(\mathcal{H}(q)) = H_M \left(\sum_{m,n \in \mathbb{Z}} (* \cdots *) \right) = \sum_{s,t \in \mathbb{Z}} (* \cdots *).$$

Cancelation under the action of the huffing operator

Theorem (C. & Tang (2022))

Let $d^* = \gcd(u, v, M)$, $d = \gcd(u, v)$, $d_u = \gcd(u, M)$ and $d_v = \gcd(v, M)$. Assume that

$$\begin{cases} d^* \mid w, \\ d_u(Av^2 + Bu^2) \mid Av \cdot dM, \\ d_v(Av^2 + Bu^2) \mid Bu \cdot dM. \end{cases}$$

Assume also that

$$(-1)^{\frac{-v\kappa+u\lambda}{d}} = -1.$$

If there exists an integer J such that

$$\begin{cases} 2d(Av^2 + Bu^2) \mid (2dMAv \cdot J - 2dAvw - dAuv + dBu^2 \\ \quad + 2dA'uv - 2dB'u^2 - Auv^2 - Bu^3), \\ 2d(Av^2 + Bu^2) \mid (2dMBu \cdot J - 2dBuw + dAv^2 - dBuv \\ \quad - 2dA'v^2 + 2dB'uv + Av^3 + Bu^2v), \end{cases}$$

then

$$H_M(\mathcal{H}(q)) = 0.$$

Cancelation under the action of the huffing operator

- Consider two companions of $\mathcal{H}(q)$:

$$\begin{aligned}\mathcal{H}(q) := & q^w f\left((-1)^\kappa q^{u+A'M}, (-1)^\kappa q^{-u+(A-A')M}\right) \\ & \times f\left((-1)^\lambda q^{\nu+B'M}, (-1)^\lambda q^{-\nu+(B-B')M}\right),\end{aligned}$$

$$\begin{aligned}\hat{\mathcal{H}}(q) := & q^{\hat{w}} f\left((-1)^\kappa q^{u+A'M}, (-1)^\kappa q^{-u+(A-A')M}\right) \\ & \times f\left((-1)^\lambda q^{\nu+(B-B')M}, (-1)^\lambda q^{-\nu+B'M}\right),\end{aligned}$$

$$\check{\mathcal{H}}(q) := q^{\check{w}} f\left((-1)^\kappa q^{u+(A-A')M}, (-1)^\kappa q^{-u+A'M}\right) \\ \times f\left((-1)^\lambda q^{\nu+(B-B')M}, (-1)^\lambda q^{-\nu+B'M}\right).$$

Cancelation under the action of the huffing operator

Choose $\hat{w} = w + \left(1 - \frac{2B'}{B}\right)v$ provided that $B \mid 2B'v$.

Theorem (C. & Tang (2022))

Let $d^* = \gcd(u, v, M)$, $d = \gcd(u, v)$, $d_u = \gcd(u, M)$ and $d_v = \gcd(v, M)$. If w is a multiple of d^* , we further let K be such that $\begin{cases} K \equiv 1 \pmod{M/d^*}, \\ K \equiv 0 \pmod{d/\gcd(u, v, w)}. \end{cases}$ Assume that

$$\begin{cases} d^*B \mid 2B'v, \\ d_u(Av^2 + Bu^2) \mid Av \cdot dM, \\ d_v(Av^2 + Bu^2) \mid Bu \cdot dM, \\ (Av^2 + Bu^2) \mid (A - 2A')v^2 - (B - 2B')uv - 2Buw \cdot K, \\ B(Av^2 + Bu^2) \mid B(A - 2A')uv + A(B - 2B')v^2 + 2ABvw \cdot K. \end{cases}$$

Then

$$H_M(\mathcal{H}(q)) = (-1)^\epsilon H_M(\hat{\mathcal{H}}(q)),$$

where

$$\epsilon = \kappa \cdot \frac{(A - 2A')v^2 - (B - 2B')uv - 2Buw \cdot K}{Av^2 + Bu^2} - \lambda \cdot \frac{B(A - 2A')uv + A(B - 2B')v^2 + 2ABvw \cdot K}{B(Av^2 + Bu^2)}.$$

Cancelation under the action of the huffing operator

Choose $\check{w} = w + \left(1 - \frac{2A'}{A}\right)u + \left(1 - \frac{2B'}{B}\right)v$ provided that $AB \mid 2(A'Bu + AB'v)$.

Theorem (C. & Tang (2022))

Let $d^* = \gcd(u, v, M)$, $d = \gcd(u, v)$, $d_u = \gcd(u, M)$ and $d_v = \gcd(v, M)$. If w is a multiple of d^* , we further let K be such that $\begin{cases} K \equiv 1 \pmod{M/d^*}, \\ K \equiv 0 \pmod{d/\gcd(u, v, w)}. \end{cases}$ Assume that

$$\begin{cases} d^*AB \mid 2(A'Bu + AB'v), \\ d_u(Av^2 + Bu^2) \mid Av \cdot dM, \\ d_v(Av^2 + Bu^2) \mid Bu \cdot dM, \\ A(Av^2 + Bu^2) \mid B(A - 2A')u^2 + A(B - 2B')uv + 2ABuw \cdot K, \\ B(Av^2 + Bu^2) \mid B(A - 2A')uv + A(B - 2B')v^2 + 2ABvw \cdot K. \end{cases}$$

Then

$$H_M(\mathcal{H}(q)) = (-1)^\varepsilon H_M(\check{\mathcal{H}}(q)),$$

where

$$\varepsilon = -\kappa \cdot \frac{B(A-2A')u^2 + A(B-2B')uv + 2ABuw \cdot K}{A(Av^2 + Bu^2)} - \lambda \cdot \frac{B(A-2A')uv + A(B-2B')v^2 + 2ABvw \cdot K}{B(Av^2 + Bu^2)}.$$

An Example

- **Target:** Prove that if $M = 2\ell + 6m + 3$ and $\sigma = -(2\ell + 4m + 2)k$, then for $\kappa \in \{0, 1\}$, $\lambda \in \{0, 1\}$ and any k such that $\gcd(k, M) = 1$,

$$H_M \left(q^\sigma \cdot f((-1)^\kappa q^k, (-1)^\kappa q^{\mu M - k})^{2\ell} \right. \\ \left. \times \left(\frac{f(-q^{2k}, -q^{\mu M - 2k})}{f((-1)^\lambda q^k, (-1)^\lambda q^{\mu M - k})} \right)^{2m+1} \right) = 0.$$

- *Generic Case.*

$$q^\sigma f((-1)^\kappa q^{ak}, (-1)^\kappa q^{-ak + M\mu})^\ell \left(\frac{f(-q^{2k}, -q^{-2k + M\mu})}{f((-1)^\lambda q^k, (-1)^\lambda q^{-k + M\mu})} \right)^m.$$

Set $(\ell, m) \mapsto (2\ell, 2m + 1)$ and $a = 1$.

An Example

Expanding and pairing powers of theta series.



$$f((-1)^\kappa q^{ak}, (-1)^\kappa q^{-ak+M\mu})^\ell = \mathcal{A}_0 \mathcal{F}_0 + \sum (\mathcal{A}_I + (-1)^{\kappa\ell} \mathcal{A}_{II}) \mathcal{F},$$

where each \mathcal{F}_* is a series in q^M , and

$$\mathcal{A}_0 = f((-1)^{\kappa\ell} q^{ak\ell}, (-1)^{\kappa\ell} q^{-ak\ell+M\mu\ell}),$$

while $\mathcal{A}_I, \mathcal{A}_{II}$ are of the form

$$\mathcal{A}_I := q^{a\xi k} f((-1)^{\kappa\ell} q^{ak\ell+M\mu\xi}, (-1)^{\kappa\ell} q^{-ak\ell+M\mu(\ell-\xi)}),$$

$$\mathcal{A}_{II} := q^{a(\ell-\xi)k} f((-1)^{\kappa\ell} q^{ak\ell+M\mu(\ell-\xi)}, (-1)^{\kappa\ell} q^{-ak\ell+M\mu\xi}).$$

An Example

Expanding and pairing powers of theta series.



$$\left(\frac{f(-q^{2k}, -q^{-2k+M\mu})}{f((-1)^\lambda q^k, (-1)^\lambda q^{-k+M\mu})} \right)^m = \sum (\mathcal{B}_I + (-1)^{(\lambda+1)m} \mathcal{B}_{II}) \mathcal{G},$$

where each $\mathcal{G}_*^{(*)}$ is a series in q^M , and $\mathcal{B}_I, \mathcal{B}_{II}$ are of the form

$$\mathcal{B}_I := q^{\tau k} f((-1)^{\lambda m} q^{3km+M\mu(m+\tau)}, (-1)^{\lambda m} q^{-3km+M\mu(2m-\tau)}),$$

$$\mathcal{B}_{II} := q^{(m-\tau)k} f((-1)^{\lambda m} q^{3km+M\mu(2m-\tau)}, (-1)^{\lambda m} q^{-3km+M\mu(m+\tau)}).$$

An Example

- WANT to prove

$$H_M \left(q^\sigma f((-1)^\kappa q^{ak}, (-1)^\kappa q^{-ak+M\mu})^\ell \left(\frac{f(-q^{2k}, -q^{-2k+M\mu})}{f((-1)^\lambda q^k, (-1)^\lambda q^{-k+M\mu})} \right)^m \right) = 0,$$

or

$$H_M \left(q^\sigma \left(\mathcal{A}_0 \mathcal{F}_0 + \sum (\mathcal{A}_I + (-1)^{\kappa\ell} \mathcal{A}_{II}) \mathcal{F} \right) \left(\sum (\mathcal{B}_I + (-1)^{(\lambda+1)m} \mathcal{B}_{II}) \mathcal{G} \right) \right) = 0.$$

- NEED to prove

$$H_M(q^\sigma \mathcal{A}_0 \mathcal{B}_I) = -(-1)^{(\lambda+1)m} H_M(q^\sigma \mathcal{A}_0 \mathcal{B}_{II}),$$

and

$$H_M(q^\sigma \mathcal{A}_I \mathcal{B}_I) = -(-1)^{\kappa\ell+(\lambda+1)m} H_M(q^\sigma \mathcal{A}_{II} \mathcal{B}_{II}).$$

Upon setting $\tau \mapsto m - \tau$ in the above, we automatically have

$$H_M(q^\sigma \mathcal{A}_I \mathcal{B}_{II}) = -(-1)^{\kappa\ell+(\lambda+1)m} H_M(q^\sigma \mathcal{A}_{II} \mathcal{B}_I).$$

An Example

Cancelation under the action of the huffing operator.

- Use the First Cancelation Theorem with

$$\mathcal{H}(q) = q^\sigma \mathcal{A}_0 \mathcal{B}_I,$$
$$\hat{\mathcal{H}}(q) = q^\sigma \mathcal{A}_0 \mathcal{B}_{II}.$$

- Take

κ	\mapsto	$(2\ell)\kappa$	λ	\mapsto	$(2m+1)\lambda$
u	\mapsto	$(2\ell)k$	v	\mapsto	$3(2m+1)k$
A	\mapsto	$(2\ell)\mu$	B	\mapsto	$3(2m+1)\mu$
A'	\mapsto	0	B'	\mapsto	$(2m+1+\tau)\mu$
M	\mapsto	$2\ell + 6m + 3$			
w	\mapsto	$-(2\ell + 4m + 2)k + \tau k$			
\hat{w}	$=$	$-(2\ell + 4m + 2)k + (2m + 1 - \tau)k$			

- NEED to prove

$$H_M(q^\sigma \mathcal{A}_0 \mathcal{B}_I) = -(-1)^{\lambda+1} H_M(q^\sigma \mathcal{A}_0 \mathcal{B}_{II}).$$

An Example

Cancelation under the action of the huffing operator.

- Let

$$d_0 = \gcd(2\ell, 3(2m+1)).$$

Then

$$d = \gcd(u, v) = d_0 k.$$

Also, noticing that $M = (2\ell) + 3(2m+1)$ and that k is coprime to M , we have

$$d^* = \gcd(u, v, M) = d_0,$$

$$d_u = \gcd(u, M) = d_0,$$

$$d_v = \gcd(v, M) = d_0.$$

We compute that

$$\frac{Av \cdot dM}{d_u(Av^2 + Bu^2)} = 1,$$

$$\frac{Bu \cdot dM}{d_v(Av^2 + Bu^2)} = 1.$$

An Example

Cancelation under the action of the huffing operator.

- Choosing $K = 1$ gives

$$\frac{(A - 2A')v^2 - (B - 2B')uv - 2Buw \cdot K}{Av^2 + Bu^2} = 2,$$

$$\frac{B(A - 2A')uv + A(B - 2B')v^2 + 2ABvw \cdot K}{B(Av^2 + Bu^2)} = -1.$$

- All divisibility conditions are satisfied!
- Finally, we compute that

$$\epsilon = (4\ell)\kappa + (2m + 1)\lambda.$$

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$$H_M(\mathcal{H}(q)) = (-1)^\lambda H_M(\hat{\mathcal{H}}(q)).$$

-

$$\begin{aligned} H_M(q^\sigma \mathcal{A}_0 \mathcal{B}_I) + (-1)^{\lambda+1} H_M(q^\sigma \mathcal{A}_0 \mathcal{B}_{II}) &= H_M(\mathcal{H}(q)) + (-1)^{\lambda+1} H_M(\hat{\mathcal{H}}(q)) \\ &= 0. \end{aligned}$$

Outlook

- **Other divisibility criteria?** For $M = 4\ell + 6m + 8$, $\sigma = -(2\ell + 2m + 3)k$, $\kappa \in \{1\}$, $\lambda \in \{0, 1\}$ and any k such that $\gcd(k, M) = 1$,

$$\begin{aligned} H_M & \left(q^\sigma \cdot f((-1)^\kappa q^{2k}, (-1)^\kappa q^{\mu M - 2k})^{2\ell+1} \right. \\ & \times \left. \left(\frac{f(-q^{2k}, -q^{\mu M - 2k})}{f((-1)^\lambda q^k, (-1)^\lambda q^{\mu M - k})} \right)^{4m+4} \right) \stackrel{?}{=} 0. \end{aligned}$$

Outlook

- Schröter–Mc Laughlin type expansion formula for other q -series?

Richmond and Szekeres (1978) proved that if

$$\sum_{n \geq 0} g(n)q^n := \frac{(q, q^7; q^8)_\infty}{(q^3, q^5; q^8)_\infty},$$

then

$$g(4n + 2) = 0.$$

This result cannot be directly fit into our framework as we do not find a Schröter–Mc Laughlin type expansion formula for the reciprocal of a generic theta power. (**Note.** One known approach, like Andrews and Bressoud had applied, is based on Ramanujan's ${}_1\psi_1$ formula to rewrite the series into a certain summation form with finitely many summands.)

Thank You!