On some new “multi-sum = product” identities

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Based on joint work with K. Baker, M. C. Russell and C. Sadowski

`arXiv:2208.14581`

Preliminaries

Pochhammer symbols

$$
\begin{aligned}
(a;q)_j &= (1-a)(1-aq)\dots(1-aq^{j-1}) \\
(q;q)_j &= (1-q)(1-q^2)\dots(1-q^j) \\
(a;q)_\infty &= (1-a)(1-aq)\dots \\
(a_1,a_2,a_3,\dots;q)_t &= (a_1;q)_t(a_2;q)_t(a_3;q)_t\cdots.
\end{aligned}
$$

$$1/(q;q)_n = 0 \quad (n<0)$$

$$(a;q)_n = (a)_n$$

Rogers–Ramanujan identities

RR 1

The number of partitions of n where adjacent parts differ by at least 2

$=$

The number of partitions of n where each part is $\equiv \pm 1 \pmod 5$.

Example

$$
\begin{aligned}
9 &= 9 \\
&= 1+8 \\
&= 2+7 \\
&= 3+6 \\
&= 1+3+5
\end{aligned}
\qquad
\begin{aligned}
9 &= 9 \\
&= 1+1+1+6 \\
&= 1+4+4 \\
&= 1+1+1+1+1+4 \\
&= 1+1+1+1+1+1+1+1+1
\end{aligned}
$$

RR 2

The number of partitions of n where adjacent parts differ by at least 2 and where the smallest part is at least 2

$=$

The number of partitions of n where each part is $\equiv \pm 2 \pmod 5$.

Example

$$\begin{aligned} 9 &= 9 \\ &= 2+7 \\ &= 3+6 \end{aligned} \qquad \begin{aligned} 9 &= 9 \\ &= 3+3+3 \\ &= 2+2+2+3 \end{aligned}$$

Generating functions

RR 1 $$\sum_{n\geq 0} \frac{q^{n^2}}{(q)_n} = (q, q^4; q^5)_\infty^{-1}.$$

RR 2 $$\sum_{n\geq 0} \frac{q^{n^2+n}}{(q)_n} = (q^2, q^3; q^5)_\infty^{-1}.$$

Recurrence

$R(x,q) = \sum_{i,n\geq 0} r_{i,n} x^i q^n$

$r_{i,n}$ = the number of partitions of n with i parts such that difference between adjacent parts is at least 2 (RR1)

$R(xq^j, q)$ = g.f. for partitions with difference at least 2 in adjacent parts and smallest part at least $j+1$.

$R(x,q) = R(xq,q) + xqR(xq^2,q)$

Uniqueness

In $\mathbb{Z}[[x,q]]$, there exists a unique solution to:

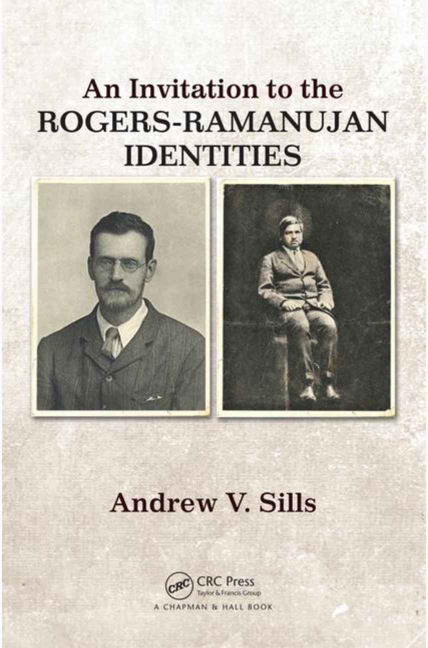
$$R(x,q) = R(xq,q) + xqR(xq^2,q)$$
$$R(0,q) = R(x,0) = 1.$$

To prove $F(q) = (q,q^4;q^5)_\infty^{-1}$,

see if there exists

$F(x,q)$

that satisfies the recurrence above.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Affine Lie algebras

There are many ways to get RR-type identities from affine Lie algebras.

Affine Lie algebras

In 80's, Lepowsky and Wilson proved RR identities using the affine Lie algebra $A_1^{(1)}$.

Lepowsky and Wilson, Meurman and Primc:

$A_1^{(1)} \leftrightsquigarrow$ Andrews–Gordon and Andrews–Bressoud identities.

What about other Lie algebras?

$A_2^{(2)}, A_\ell^{(1)}, B_\ell^{(1)}, C_\ell^{(1)}, D_\ell^{(1)}, E_{6,7,8}^{(1)}, F_4^{(1)}, G_2^{(1)}, A_{2\ell}^{(2)}, A_{2\ell-1}^{(2)}, D_{\ell+1}^{(2)}, E_6^{(2)}, D_4^{(3)}.$

Next in line: $A_2^{(2)}$

RR-type identities

1. Capparelli identities (1988)

Level 3 modules for $A_2^{(2)}$

Two identities

Capparelli identities (1988)

C1

Number of partitions of n where

2 does not appear as a part,

the difference of two consecutive parts is at least 2, and

is exactly 2 or 3 only if their sum is a multiple of 3.

=

Coefficient of q^n in $(-q, -q^3, -q^5, -q^6; q^6)_\infty$.

Proofs

By Andrews, Alladi–Andrews–Gordon, Tamba–Xie, Capparelli, etc.

Generating functions

$$\sum_{i,j\geq 0} \frac{q^{2i^2+6ij+6j^2}}{(q)_i(q^3;q^3)_j} = (-q,-q^3,-q^5,-q^6;q^6)_\infty$$

Surprisingly recent!

Independently due to K.–Russell (2019) and Kurşungöz (2019)

2. Nandi identities (2014), Proof by Takigiku–Tsuchioka (2019)

Level 4 modules for $A_2^{(2)}$

Three identities

M. Takigiku and S. Tsuchioka, *A proof of conjectured partition identities of Nandi*, Amer. J. Math, to appear, `arXiv:1910.12461`.

A partition $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_s > 0$ satisfies difference condition $[d_1, d_2, \ldots, d_s]$ if $\lambda_i - \lambda_{i+1} = d_i$ for all $1 \leq i \leq s-1$.

For example:

$8+7+6+5$ satisfies the difference condition $[1]$.

Nandi identities (2014), Proof by Takigiku–Tsuchioka (2019)

N1

Number of partitions of n into parts different from 1 such that there is no contiguous sub-partition satisfying the difference conditions $[1]$, $[0,0]$, $[0,2]$, $[2,0]$ or $[0,3]$, and such that there is no sub-partition with an odd sum of parts satisfying the difference conditions $[3,0]$, $[0,4]$, $[4,0]$ or $[3,2*,3,0]$ (where $2*$ indicates zero or more occurrences of 2)

=

Coefficient of q^n in $(q^2,q^3,q^4,q^{10},q^{11},q^{12};q^{14})_\infty^{-1}$.

GULP!

Sum-sides?

N1

$$\sum_{i,j}(-1)^j \frac{q^{\binom{i}{2}+2\binom{j}{2}+2ij+i+j}}{(q)_i(q^2;q^2)_j} = (q^2,q^3,q^4,q^{10},q^{11},q^{12};q^{14})_\infty^{-1}$$

See: Takigiku–Tsuchioka

3. Some conjectures from 2014/5 (K.–Russell)

Related to level 3 modules of $D_4^{(3)}$

Three identities

(Obtained purely experimentally, without any algebra.)

Partition $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0$

has difference at least d at distance k

if $\lambda_i - \lambda_{i+k} \geq d$ for all $1 \leq i \leq s-k$.

For example, RR partitions have distance at least 2 at distance 1.

Some conjectures from 2014/5 (K.–Russell)

I 1

Number of partitions of n with difference at least 3 at distance 2 such that if two consecutive parts differ by at most 1, then their sum is divisible by 3.

$\stackrel{?}{=}$

Coefficient of q^n in $(q, q^3, q^6, q^8; q^9)_\infty^{-1}$.

Sum-sides

$$\sum_{i,j} \frac{q^{i^2+3ij+3j^2}}{(q)_i(q^3;q^3)_j} \stackrel{?}{=} (q,q^3,q^6,q^8;q^9)_\infty^{-1}.$$

Due to Kurşungöz (2019)

Recent developments

Tsuchioka's work (2022) implies:

$$\sum_{i,j} \frac{q^{i^2+3ij+3j^2}}{(q)_i(q^3;q^3)_j} \geq (q,q^3,q^6,q^8;q^9)_\infty^{-1}.$$

S. Tsuchioka, *A vertex operator reformulation of the Kanade-Russell conjecture modulo 9*, `arXiv:2211.12351`.

Relationship

These conjectures	level 3 of $D_4^{(3)}$	$\displaystyle\sum_{i,j} \frac{q^{i^2+3ij+3j^2}}{(q)_i(q^3;q^3)_j}$
Capparelli	level 3 of $A_2^{(2)}$	$\displaystyle\sum_{i,j} \frac{q^{2i^2+6ij+6j^2}}{(q)_i(q^3;q^3)_j}$

This Talk

New “sum=product” identities

1. New identities for level 4 for $D_4^{(3)}$.

2. New quadruple sum-sides for Nandi’s identities. Manifestly positive for N1.

Relationship

New identities	level 4 of $D_4^{(3)}$	
Nandi	level 4 of $A_2^{(2)}$	double the quadratic form

Everything with proof!

Sums for Nandi’s identities

Chris Sadowski Mon, Nov 25, 2019, 8:24 AM

to Matthew, me

Funny story... As I'm typing these results and redoing computations, I stumbled into this because of a typo... I accidentally doubled the exponent in the mod 10 identities and got this:

Conjecture 11.

$$\sum_{\mathbf{m}\in(\mathbb{Z}_{\geq 0})^4} \frac{q^{4m_1^2+12m_1m_2+8m_1m_3+4m_1m_4+12m_2^2+16m_2m_3+8m_2m_4+6m_3^2+6m_3m_4+2m_4^2}}{(q^2;q^2)_{m_1}(q^2;q^2)_{m_2}(q;q)_{m_3}(q;q)_{m_4}}$$

$$= \prod_{k\equiv\pm 2,\pm 3,\pm 4 \mod 14} \frac{1}{1-q^k}$$

Immediate steps

$$
\begin{aligned}
&S_{A,B,C,D}(x,q)\\
&= \sum \frac{x^{2i+3j+2k+\ell} q^{4i^2+12ij+8ik+4i\ell+12j^2+16jk+8j\ell+6k^2+6k\ell+2\ell^2+Ai+Bj+Ck+D\ell}}{(q^2;q^2)_i\,(q^2;q^2)_j\,(q;q)_k\,(q;q)_\ell}.
\end{aligned}
$$

Conjecture

$$
\begin{aligned}
N_1(x,q) &= S_{0,0,0,0}(x,q),\\
N_2(x,q) &= S_{0,-2,-2,-1}(x,q) - S_{0,0,0,0}(x,q) + S_{2,2,1,0}(x,q),\\
N_3(x,q) &= \frac{q^2}{x^2} S_{-8,-12,-8,-4}(x,q) - \frac{1}{x} S_{-2,-4,-3,-2}(x,q) - \frac{1}{q^2} S_{0,0,0,0}(x,q)\\
&\quad - \frac{q^2}{x^2} S_{-4,-8,-6,-3}(x,q).
\end{aligned}
$$

Proof: Broad steps

1. Takigiku–Tsuchioka provide a system of difference equations satisfied by the (x, q) generating functions in Nandi's identities.
2. Show that our conjectural formulas satisfy this system of difference equations.

Requires a non-trivial amount of computer assistance.

Takigiku–Tsuchioka system

$$\begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 = N_2 \\ F_4 = N_3 \\ F_5 \\ F_7 = N_1 \end{bmatrix} = \begin{bmatrix} 1 & xq^2 & x^2q^4 & xq & x^2q^2 & 0 & 0 \\ 0 & xq^2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & xq^2 & 0 & xq & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & xq^2 & 0 & 1 \\ 1 & xq^2 & x^2q^4 & xq & 0 & 0 & 0 \\ 1 & xq^2 & x^2q^4 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} F_0(xq^2,q) \\ F_1(xq^2,q) \\ F_2(xq^2,q) \\ F_3(xq^2,q) \\ F_4(xq^2,q) \\ F_5(xq^2,q) \\ F_7(xq^2,q) \end{bmatrix}$$

Conjecture

$$F_7(x,q) = S_{0,0,0,0}(x,q),$$
$$F_1(x,q) = S_{2,2,1,0}(x,q),$$
$$F_5(x,q) = S_{0,-2,-2,-1}(x,q).$$

We have:

$$F_2(x,q) = F_7(xq^2,q),$$
$$F_3(x,q) = F_1(x,q) + F_5(x,q) - F_7(x,q),$$
$$F_0(x,q) = F_7(xq^{-2},q) - xF_1(x,q) - x^2F_2(x,q),$$
$$F_4(x,q) = x^{-2}q^2F_0(xq^{-2},q) - x^{-2}q^2F_5(xq^{-2},q).$$

Modified Murray–Miller algorithm

System of difference equations

$\rightsquigarrow$

higher order difference equations satisfied by F_i.

Takigiku–Tsuchioka's paper / Andrews' Chapter 8

Higher order difference equations

The power series $F_7 = N_1$ is the unique solution in $\mathbb{Z}[[x,q]]$ of:

$$\begin{aligned}
0 = {} & F_7(x,q) \\
& + (-xq^4 - xq^3 - xq^2 - 1)F_7(xq^2,q) \\
& + (xq^5 + xq^4 + xq^3 - x + 1)q^4 x F_7(xq^4,q) \\
& - x^2q^6(xq^9 - xq^6 - xq^5 - xq^4 + 1)F_7(xq^6,q) \\
& - x^3q^{13}(xq^8 + xq^7 + xq^6 - q^2 - q - 1)F_7(xq^8,q) \\
& + x^3q^{17}(x^2q^{14} - xq^8 - xq^6 + 1)F_7(xq^{10},q),
\end{aligned}$$

$$1 = F_7(x,0) = F_7(0,q).$$

We need to show that $S_{0,0,0,0}(x,q)$ satisfies this.

To show

$$S_{A,B,C,D}(x,q) = \sum \frac{x^{2i+3j+2k+\ell} q^{4i^2+12ij+8ik+4i\ell+12j^2+16jk+8j\ell+6k^2+6k\ell+2\ell^2+Ai+Bj+Ck+D\ell}}{(q^2;q^2)_i (q^2;q^2)_j (q;q)_k (q;q)_\ell}.$$

Then:

$$\begin{aligned} 0 =& S_{0,0,0,0} \\ &+(-xq^4-xq^3-xq^2-1)S_{4,6,4,2} \\ &+q^4x(xq^5+xq^4+xq^3-x+1)S_{8,12,8,4} \\ &-x^2q^6(xq^9-xq^6-xq^5-xq^4+1)S_{12,18,12,6} \\ &-x^3q^{13}(xq^8+xq^7+xq^6-q^2-q-1)S_{16,24,16,8} \\ &+x^3q^{17}(x^2q^{14}-xq^8-xq^6+1)S_{20,30,20,10} \end{aligned}$$

The Key

S series satisfy a few easily deduced “atomic” relations.

We show: required relation is a consequence of the atomic relations.

Requires highly non-trivial computer assistance.

Atomic relations

Recall:

$$S_{A,B,C,D}(x,q) = \sum \frac{x^{2i+3j+2k+\ell} q^{4i^2+12ij+8ik+4i\ell+12j^2+16jk+8j\ell+6k^2+6k\ell+2\ell^2+Ai+Bj+Ck+D\ell}}{(q^2;q^2)_i (q^2;q^2)_j (q;q)_k (q;q)_\ell}.$$

Then:

$$S_{A,B,C,D} - S_{A+2,B,C,D} = x^2 q^{4+A} S_{A+8,B+12,C+8,D+4},$$
$$S_{A,B,C,D} - S_{A,B+2,C,D} = x^3 q^{12+B} S_{A+12,B+24,C+16,D+8},$$
$$S_{A,B,C,D} - S_{A,B,C+1,D} = x^2 q^{6+C} S_{A+8,B+16,C+12,D+6},$$
$$S_{A,B,C,D} - S_{A,B,C,D+1} = x q^{2+D} S_{A+4,B+8,C+6,D+4}.$$

A slight modification

$$n_1(A,B,C,D): S_{A,B,C,D} - S_{A+4,B,C,D} - x^2 q^{4+A}(1+q^2) S_{A+8,B+12,C+8,D+4} + x^4 q^{18+2A} S_{A+16,B+24,C+16,D+8} = 0,$$

$$n_2(A,B,C,D): S_{A,B,C,D} - S_{A,B+2,C,D} - x^3 q^{12+B} S_{A+12,B+24,C+16,D+8} = 0,$$

$$n_3(A,B,C,D): S_{A,B,C,D} - S_{A,B,C+2,D} - x^2 q^{6+C} S_{A+8,B+16,C+12,D+6} - x^2 q^{7+C} S_{A+8,B+16,C+12,D+6} + x^4 q^{25+2C} S_{A+16,B+32,C+24,D+12} = 0,$$

$$n_4(A,B,C,D): S_{A,B,C,D} - S_{A,B,C,D+2} - x q^{2+D} S_{A+4,B+8,C+6,D+4} - x q^{3+D} S_{A+4,B+8,C+6,D+4} + x^2 q^{9+2D} S_{A+8,B+16,C+12,D+8} = 0.$$

Computer Assistance

Show that for a large enough K,
the required relation is a linear combination over $\mathbb{Q}(z,q)$ of
$\{n_i(A,B,C,D) \mid 1 \leq i \leq 4,\ -K \leq A,B,C,D \leq K\}$.

How?

Setup a large system of unknowns and solve it.

The proof itself is long and computationally intensive to find, but,

It takes only seconds to check it.

Proof certificate

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```
N1(0,-4,-4,2)/x^2/q^10*(q^6*x-1)/(q^2+1)*(q^39*x^2+10*q^38*x^2+49*q^37*x^2-q^37*x+158*q^36*x^2-8
*q^36*x+381*q^35*x^2-34*q^35*x+735*q^34*x^2-q^35-103*q^34*x+1178*q^33*x^2-4*q^34-245*q^33*x+1595
*q^32*x^2-7*q^33-485*q^32*x+1810*q^31*x^2-6*q^32-832*q^31*x+1621*q^30*x^2-3*q^31-1261*q^30*x+847
*q^29*x^2-14*q^30-1698*q^29*x-629*q^28*x^2-67*q^29-2020*q^28*x-2818*q^27*x^2-195*q^28-2063*q^27*
x-5590*q^26*x^2-387*q^27-1655*q^26*x-8671*q^25*x^2-602*q^26-640*q^25*x-11684*q^24*x^2-764*q^25+1
059*q^24*x-14226*q^23*x^2-796*q^24+3401*q^23*x-15952*q^22*x^2-633*q^23+6224*q^22*x-16644*q^21*x^
2-257*q^22+9215*q^21*x-16244*q^20*x^2+313*q^21+11958*q^20*x-14861*q^19*x^2+999*q^20+13997*q^19*x
-12752*q^18*x^2+1706*q^19+15058*q^18*x-10265*q^17*x^2+2328*q^18+14997*q^17*x-7754*q^16*x^2+2794*
q^17+13896*q^16*x-5495*q^15*x^2+3010*q^16+11941*q^15*x-3646*q^14*x^2+2947*q^15+9520*q^14*x-2258*
q^13*x^2+2630*q^14+7010*q^13*x-1310*q^12*x^2+2165*q^13+4782*q^12*x-733*q^11*x^2+1651*q^12+3023*q
^11*x-425*q^10*x^2+1175*q^11+1796*q^10*x-274*q^9*x^2+785*q^10+1025*q^9*x-189*q^8*x^2+501*q^9+595
*q^8*x-122*q^7*x^2+320*q^8+376*q^7*x-65*q^6*x^2+220*q^7+263*q^6*x-26*q^5*x^2+164*q^6+188*q^5*x-7
*q^4*x^2+118*q^5+122*q^4*x-q^3*x^2+74*q^4+65*q^3*x+38*q^3+26*q^2*x+16*q^2+7*q*x+5*q+x+1)/(q^4+1)
/(q^12*x+2*q^11*x+2*q^10*x-q^10+2*q^9*x-2*q^9+2*q^8*x-2*q^8+2*q^7*x-2*q^7+q^6*x-q^6+q^4*x+2*q^3*
x+q^2*x-q^2-2*q-1)/(q^9-q^6-q^4-q^2-q-1)/(q^3+q^2+q+1)/(q+1)
+N1(0,-4,2,2)/x^3/q*(q^6*x-1)/(q^2+1)*(q^7+q^6-1)*(q^24*x^2+4*q^23*x^2+8*q^22*x^2+11*q^21*x^2-q^
21*x+11*q^20*x^2-4*q^20*x+8*q^19*x^2-8*q^19*x+2*q^18*x^2-10*q^18*x-8*q^17*x^2-14*q^17*x-21*q^16*
x^2-3*q^17-14*q^16*x-35*q^15*x^2-4*q^16-14*q^15*x-49*q^14*x^2-5*q^15-4*q^14*x-62*q^13*x^2-4*q^14
+6*q^13*x-70*q^12*x^2-3*q^13+23*q^12*x-71*q^11*x^2+q^12+39*q^11*x-66*q^10*x^2+4*q^11+57*q^10*x-5
6*q^9*x^2+6*q^10+69*q^9*x-44*q^8*x^2+10*q^9+75*q^8*x-33*q^7*x^2+13*q^8+74*q^7*x-23*q^6*x^2+15*q^
7+66*q^6*x-13*q^5*x^2+13*q^6+52*q^5*x-5*q^4*x^2+11*q^5+37*q^4*x-q^3*x^2+8*q^4+24*q^3*x+6*q^3+13*
q^2*x+3*q^2+5*q*x+q+x)/(q^7-q^5-q^4+q^3-q-1)/(q^12+q^10-q^9-q^8-q^7-q^6+q^4+q^2+1)/(q^6+q^4+q^2+
1)/(q+1)
-N1(0,-2,-4,-2)*(q^54*x^3+10*q^53*x^3+51*q^52*x^3-2*q^52*x^2+175*q^51*x^3-18*q^51*x^2+456*q^50*x
^3-87*q^50*x^2+961*q^49*x^3-292*q^49*x^2+1697*q^48*x^3+4*q^49*x-756*q^48*x^2+2557*q^47*x^3+27*q^
```

Line 1, Column 1 Tab Size: 4 Plain Text

Theorem (Baker–K.–Russell–Sadowski, 2022)

$$F_7(x,q) = S_{0,0,0,0}(x,q),$$
$$F_1(x,q) = S_{2,2,1,0}(x,q),$$
$$F_5(x,q) = S_{0,-2,-2,-1}(x,q).$$

Theorem (Baker–K.–Russell–Sadowski, 2022)

$$\begin{aligned}
N_1(x,q) &= S_{0,0,0,0}(x,q),\\
N_2(x,q) &= S_{0,-2,-2,-1}(x,q) - S_{0,0,0,0}(x,q) + S_{2,2,1,0}(x,q),\\
N_3(x,q) &= \frac{q^2}{x^2}S_{-8,-12,-8,-4}(x,q) - \frac{1}{x}S_{-2,-4,-3,-2}(x,q) - \frac{1}{q^2}S_{0,0,0,0}(x,q)\\
&\quad - \frac{q^2}{x^2}S_{-4,-8,-6,-3}(x,q).
\end{aligned}$$

Mod-10 identities

Mod-10 identities

$$\sum \frac{q^{2i^2+6ij+4ik+2i\ell+6j^2+8jk+4j\ell+3k^2+3k\ell+\ell^2}}{(q^2;q^2)_i\,(q^2;q^2)_j\,(q;q)_k\,(q;q)_\ell} = \frac{1}{(q,q^4;q^5)_\infty(q^2,q^8;q^{10})_\infty}$$

+ 3 more.

$$R_{A,B,C,D}(x,y,q) = \sum \frac{x^{2j+k+\ell} y^{i+j+k} q^{2i^2+6ij+4ik+2i\ell+6j^2+8jk+4j\ell+3k^2+3k\ell+\ell^2+Ai+Bj+Ck+D\ell}}{(q^2;q^2)_i (q^2;q^2)_j (q;q)_k (q;q)_\ell}.$$

Conjecture

$$R_{0,0,0,0}(1,1,q) = \frac{1}{(q,q^4;q^5)_\infty (q^2,q^8;q^{10})_\infty},$$

$$R_{0,0,0,0}(q,1,q) = \frac{1}{(q^2,q^3;q^5)_\infty (q^2,q^8;q^{10})_\infty},$$

$$R_{0,0,0,0}(1,q^2,q) = \frac{1}{(q,q^4;q^5)_\infty (q^4,q^6;q^{10})_\infty},$$

$$R_{0,0,0,0}(q,q^2,q) = \frac{1}{(q^2,q^3;q^5)_\infty (q^4,q^6;q^{10})_\infty}.$$

To show

$R_{0,0,0,0}(x,y,q)$ satisfies:

$$\begin{aligned} F(x,y,q) &= F(xq,y,q) + xqF(xq^2,y,q), \\ F(x,y,0) &= 1, \\ F(x,0,q) &= \sum_{\ell} \frac{q^{\ell^2} x^{\ell}}{(q)_{\ell}}, \\ F(0,y,q) &= \sum_{i} \frac{q^{2i^2} y^{i}}{(q^2;q^2)_{i}}. \end{aligned}$$

This system has a unique solution in $\mathbb{Z}[[x,y,q]]$:

$$F(x,y,q) = \left(\sum_{\ell} \frac{q^{\ell^2} x^{\ell}}{(q)_{\ell}} \right) \left(\sum_{i} \frac{q^{2i^2} y^{i}}{(q^2;q^2)_{i}} \right)$$

Atomic relations

$$
\begin{aligned}
m_1(A,B,C,D): &\quad R_{A,B,C,D} - R_{A+2,B,C,D} - yq^{2+A} R_{A+4,B+6,C+4,D+2} &= 0,\\
m_2(A,B,C,D): &\quad R_{A,B,C,D} - R_{A,B+2,C,D} - x^2 yq^{6+B} R_{A+6,B+12,C+8,D+4} &= 0,\\
m_3(A,B,C,D): &\quad R_{A,B,C,D} - R_{A,B,C+1,D} - xyq^{3+C} R_{A+4,B+8,C+6,D+3} &= 0,\\
m_4(A,B,C,D): &\quad R_{A,B,C,D} - R_{A,B,C,D+1} - xq^{1+D} R_{A+2,B+4,C+3,D+2} &= 0.
\end{aligned}
$$

Proof

Proving $F(x,y,q) = F(xq,y,q) + xqF(xq^2,y,q)$ for $R_{0,0,0,0}(x,y,q)$

Translates to proving $R_{0,0,0,0} - R_{0,2,1,1} - xqR_{0,4,2,2} = 0$.

This can be obtained as:

$$
\begin{aligned}
&-m_1(-2,0,0,0) + m_1(-2,0,0,1) - xq \cdot m_1(0,4,2,2) \\
&+ xq \cdot m_1(0,4,3,2) + m_2(0,0,0,1) + m_3(0,2,0,1) \\
&- xq \cdot m_3(2,4,2,2) + m_4(-2,0,0,0) - y \cdot m_4(2,6,4,2).
\end{aligned}
$$

Final Remarks

1. Manifest positivity

There are seven interlocking series in Takigiku–Tsuchioka's proof.

$F_0,\ F_1,\ F_2,\ F_3 = N_2,\ F_4 = N_3,\ F_5,\ F_7 = N_1.$

Our identities give manifestly positive quadruple sums for:

$F_1, F_5, F_7 = N_1.$

You can find their combinatorial interpretations in our paper.

What makes these three special?

How to show explicitly that these sums count the partitions?

2. Combinatorial interpretations

For the sums in Mod-10 identities?

$$R_{A,B,C,D}(x,y,q) = \sum \frac{x^{2j+k+\ell} y^{i+j+k} q^{2i^2+6ij+4ik+2i\ell+6j^2+8jk+4j\ell+3k^2+3k\ell+\ell^2+Ai+Bj+Ck+D\ell}}{(q^2;q^2)_i (q^2;q^2)_j (q;q)_k (q;q)_\ell}.$$

3. Computer assistance

Packages by Research Institute for Symbolic Computation (RISC) may be useful.

Chern and Li: Use RISC packages

S. Chern, Z. Li, *Linked partition ideals and Kanade-Russell conjectures*, Discrete Math. 343 (2020), no. 7, 111876, 24 pp.

Chern: Non-computer assisted

S. Chern, *Linked partition ideals, directed graphs and q-multi-summations*, Electron. J. Combin. 27 (2020), no. 3, Paper No. 3.33, 29 pp.

4. More about the proof technique

Using this technique, jointly with Russell, we proved (new) Mod-6 and Mod-10 A_2 Andrews–Schilling–Warnaar identities.

See:

Matthew Russell's talk in this seminar from 2022.

S. Kanade, M. C. Russell, *Completing the A_2 Andrews–Schilling–Warnaar identities*, IMRN, to appear, `arXiv:2203.05690`.

Recently, Ali Uncu has found a much faster framework to perform the computational part.

A. K. Uncu, *Proofs of Modulo 11 and 13 Cylindric Kanade-Russell Conjectures for A_2 Rogers-Ramanujan Type Identities*, `arXiv:2301.01359`.

5. The blossoms in Ramanujan’s garden …

Many ways to get deep and interesting combinatorics from affine Lie algebras / vertex operator algebras.

S. Capparelli, A. Meurman, A. Primc, M. Primc,

J. Dousse, I. Konan + friends

O. Foda, T. Welsh

C. Sadowski (+ friends)

M. Takigiku, S. Tsuchioka, K. Ito

O. Warnaar (+ friends)

…

Thank you!