Tight Cylindric Partitions

Shashank Kanade

University of Denver

Seminar in Partition Theory, q-Series and Related Topics

Joint work with Matthew C. Russell

Partitions

Definition

Given $n \in \mathbb{Z}_{\geq 0}$,

Number of ways to write n as:

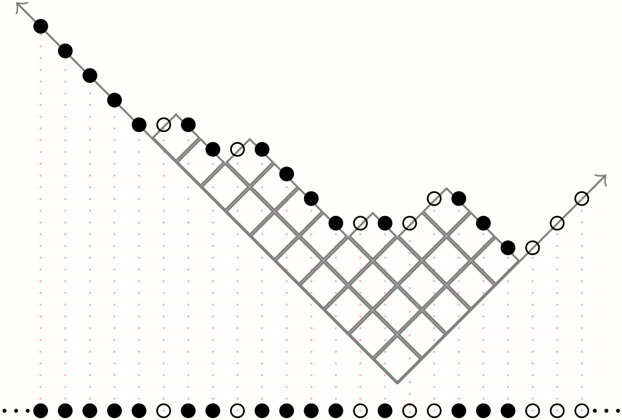
$n = \pi_1 + \cdots + \pi_t$

with $t \geq 0$, $\pi_1 \geq \pi_2 \geq \cdots \geq \pi_t \in \mathbb{Z}_{>0}$,

Example

$4 = 4, 3+1, 2+2, 2+1+1, 1+1+1+1$

Ferrer’s diagram: Russian notation



$$28 = 10 + 8 + 4 + 3 + 3$$

Maya diagram

$$28 = 10 + 8 + 4 + 3 + 3$$

$$\cdots \quad \bullet \quad \bullet \quad \circ \quad \bullet \quad \bullet \quad \circ \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \circ \quad \bullet \quad \circ \quad \circ \quad \bullet \quad \bullet \quad \bullet \quad \circ \quad \circ \quad \cdots$$

Infinite sea of $\bullet$ to the left

Infinite sea of $\circ$ to the right

Number of beads to the right of i th vacancy $=$ i th part.

Maya diagram

$$28 = 10 + 8 + 4 + 3 + 3$$

$$\cdots \;\bullet\; \bullet\; \circ\; \bullet\; \bullet\; \circ\; \bullet\; \bullet\; \bullet\; \bullet\; \circ\; \bullet\; \circ\; \circ\; \bullet\; \bullet\; \bullet\; \circ\; \circ\; \cdots$$

Infinite sea of $\bullet$ to the left

Infinite sea of $\circ$ to the right

Number of beads to the right of i th vacancy $=$ i th part.

Cylindric Partitions

Definition: Cylindric Partitions

Fix a number of rows r also called rank

Fix a profile which is a composition $c = (c_1, c_2, \cdots, c_r)$ of r many non-negative integers.

$\ell = c_1 + c_2 + \cdots + c_r$ is called the level.

Definition: Cylindric Partitions

Fix a number of rows r also called rank

Fix a profile which is a composition $c = (c_1, c_2, \cdots, c_r)$ of r many non-negative integers.

$\ell = c_1 + c_2 + \cdots + c_r$ is called the level.

Attached to this data are cylindric partitions of profile c:

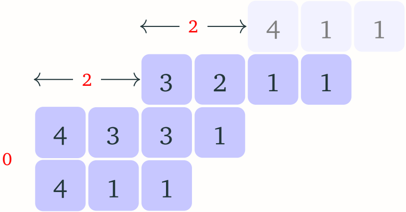
a tuple of partitions $\pi = (\pi^{(1)}, \cdots, \pi^{(r)})$ satisfying some properties

Definition by Example

$c = (2, 2, 0)$.

This has rank $r = 3$, level $\ell = 2 + 2 + 0 = 4$.

$\pi = (\pi^{(1)}, \pi^{(2)}, \pi^{(3)}) = ((3,2,1,1), (4,3,3,1), (4,1,1))$



The profile $c = (c_1, \cdots, c_r)$ gives indentations

The profile $c = (c_1, \cdots, c_r)$ gives indentations

Rank r is the number of rows, or the number of individual partitions.

The profile $c = (c_1, \cdots, c_r)$ gives indentations

Rank r is the number of rows, or the number of individual partitions.

Level is the total indentation

The profile $c = (c_1, \cdots, c_r)$ gives indentations

Rank r is the number of rows, or the number of individual partitions.

Level is the total indentation

Parts weakly decrease across rows and columns

The profile $c=(c_1,\cdots,c_r)$ gives indentations

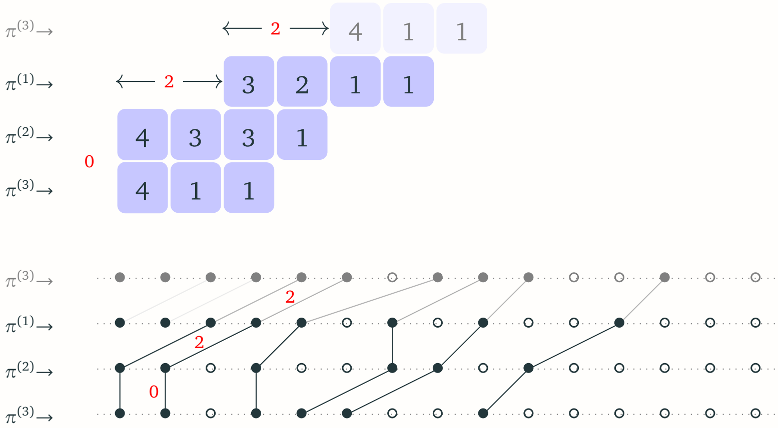
Rank r is the number of rows, or the number of individual partitions.

Level is the total indentation

Parts weakly decrease across rows and columns

Cylindric because numbers weakly decrease across columns even when the bottom row is pasted on top appropriately indented.

Abacus diagram



Abaci: Summary

Putting together the maya diagrams of individual partitions with correct indentations and then yoking the beads together.

Infinite repeating pattern of yokes far to left matches the profile c indentations

Generating functions

$$C_c(z,q) = \sum_{\pi} z^{\max(\pi)} q^{\mathrm{wt}(\pi)}$$

So, for $c = (2,2,0)$ and $\pi = ((3,2,1,1),(4,3,3,1),(4,1,1))$

$\max = 4$

$\mathrm{wt} = 3+2+1+1+4+3+3+1+4+1+1$

Some important facts

Some important facts

(a) If c' is obtained from c by rotation, then $C_c(z,q) = C_{c'}(z,q)$

Some important facts

(a) If c' is obtained from c by rotation, then $C_c(z,q) = C_{c'}(z,q)$

(b) If c' is obtained by reversing c, then $C_c(1,q) = C_{c'}(1,q)$

(c) $C_c(1,q)$ has a periodic product form

Borodin,

Kyoto school? / Tingley,

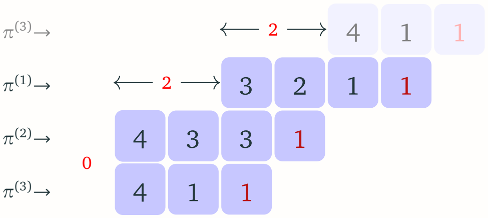
Gessel–Krattenthaler + Foda–Welsh

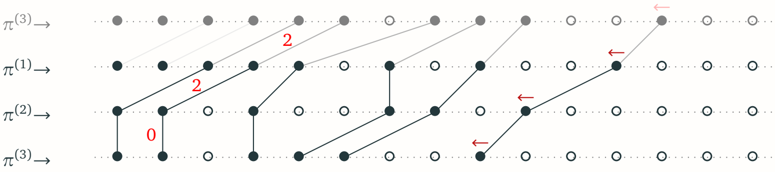
(d) In the last decade or so, there have been many studies devoted to finding closed form sum-side expressions for $C_c(z,q)$ for two-row and three-row cylindric partitions.

(d) In the last decade or so, there have been many studies devoted to finding closed form sum-side expressions for $C_c(z,q)$ for two-row and three-row cylindric partitions.

Bersakçi, Bridges, Corteel, Dousse, Foda, Kurşungöz, Langer, Li, Russell, Seyrek, Tsuchioka, Uncu, Warnaar, Welsh, ...

Tight cylindric partitions


$\pi^{(3)}\rightarrow$
$\pi^{(1)}\rightarrow$
$\pi^{(2)}\rightarrow$
$\pi^{(3)}\rightarrow$
2
4 1 1
2
3 2 1 1
4 3 3 1
0
4 1 1


$\pi^{(3)}\rightarrow$
$\pi^{(1)}\rightarrow$
$\pi^{(2)}\rightarrow$
$\pi^{(3)}\rightarrow$
2
2
0

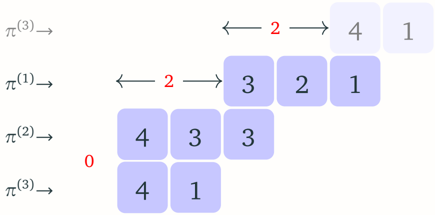
Tight cylindric partitions

A cylindric partition whose abacus diagram is as tight as possible, i.e., none of the yokes could be moved further to the left.

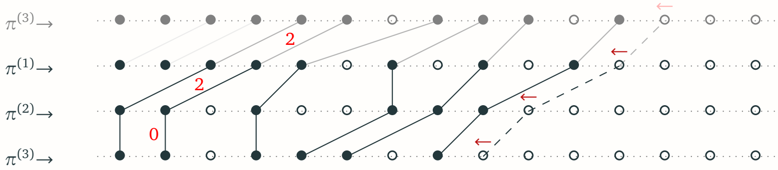
Equivalent definition

A cylindric partition $\pi = (\pi^{(1)}, \cdots, \pi^{(r)})$ of profile $c = (c_1, \cdots, c_r)$ is called tight if for every $j \in \mathbb{Z}_{>0}$, there exists a $\pi^{(i)}$ that does *not* contain j as a part.

Example



$\pi^{(3)}\rightarrow$
$\longleftarrow$ 2 $\longrightarrow$
4 1
$\pi^{(1)}\rightarrow$
$\longleftarrow$ 2 $\longrightarrow$
3 2 1
$\pi^{(2)}\rightarrow$
4 3 3
0
$\pi^{(3)}\rightarrow$
4 1



$\pi^{(3)}\rightarrow$
$\pi^{(1)}\rightarrow$
$\pi^{(2)}\rightarrow$
$\pi^{(3)}\rightarrow$
2
2
0

Generating functions

For a given profile c, we let $\mathcal{T}_c$ be the set of tight cylindric partitions of profile c.

$$T_c(z,q) = \sum_{\pi \in \mathcal{T}_c} z^{\max(\pi)} q^{\mathrm{wt}(\pi)}$$

Product form

For profile $c = (c_1, \cdots, c_r)$,

$$T_c(1,q) = \frac{(q^r;q^r)_\infty (q^m;q^m)_\infty^{r-1}}{(q)_\infty^r} \prod_{1 \le i < j \le r} \theta(q^{j-i+c_i+\cdots+c_{j-1}};q^m)$$

where,

$$m = r + \ell = r + c_1 + \cdots + c_r$$

$$\theta(a;q) = (a;q)_\infty (q/a;q)_\infty.$$

Example

If $c = (b, \ell - b)$ or $(\ell - b, b)$ for $0 \le b \le \ell$, we have:

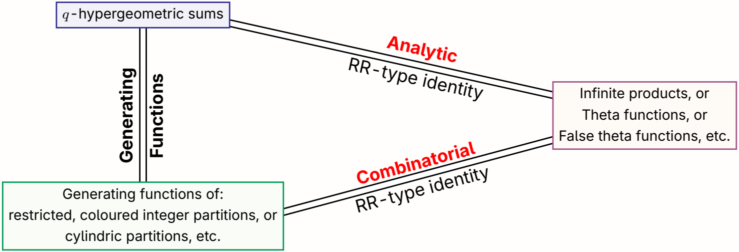
$$
\begin{aligned}
T_c(1,q) &= \frac{(q^2;q^2)_\infty (q^{\ell+2};q^{\ell+2})_\infty}{(q)_\infty^2} \theta(q^{b+1};q^{\ell+2}) \\
&= \frac{(q^{b+1}, q^{\ell-b+1}, q^{\ell+2}; q^{\ell+2})_\infty}{(q;q^2)_\infty (q)_\infty}
\end{aligned}
$$

So, upto a factor of $(q;q^2)_\infty$, these are the products in Andrews–Gordon and Andrews–Bressoud identities.

For a rank r profile c,

$T_c(1,q) = (q^r;q^r)_\infty C_c(1,q).$

The main question



What about the sums???

Functional Equations

We are looking for bivariate generating functions, i.e., $T_c(z;q)$.

We are looking for bivariate generating functions, i.e., $T_c(z;q)$.

Two broad strategies:

(1) Either do clever counting and produce closed-form generating functions

(2) Guess the formulas and prove that they satisfy required functional equations.

We are looking for bivariate generating functions, i.e., $T_c(z;q)$.

Two broad strategies:

(1) Either do clever counting and produce closed-form generating functions

(2) Guess the formulas and prove that they satisfy required functional equations.

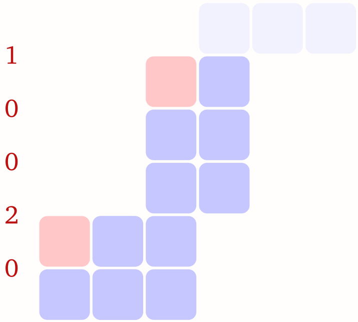
At least the two-row case ought to be manageable ....

For cylindric partitions, Corteel–Welsh showed that the generating functions are unique solutions to certain z, q functional equations.

These functional equations tie together all generating functions for c of a fixed rank and level.

Example

$c = (1, 0, 0, 2, 0)$



The maximum part must appear on a subset of red rectangles.

These red rectangles appear in rows where $c_i > 0$.

Peeling off all maximum parts leads to another cylindric partition with same level and same rank, but different profile.

Corteel–Welsh functional equations

$$C_c(z;q) = \sum_{\emptyset \neq J \subseteq I_c} (-1)^{|J|-1} \frac{C_{c(J)}(zq^{|J|};q)}{1 - zq^{|J|}}.$$

I_c = Set of red rectangles in a profile c

$c(J)$ = profile of same rank and level obtained after removing a subset of these rectangles.

Along with trivial initial conditions

$C_c(0;q) = C_c(z;0) = 1,$

Corteel–Welsh functional equations uniquely determine $C_c(z;q)$ where c runs through profiles of a fixed rank and level.

Equations for tight cylindrics

Functional equation for tight cylindric partitions are as follows.

Theorem (K.–Russell)

For c a profile of rank r,

$$T_c(z;q) + \frac{zq^r}{1-zq^r} T_c(zq^r;q) = \sum_{\emptyset \neq J \subseteq I_c} (-1)^{|J|-1} \frac{T_{c(J)}(zq^{|J|};q)}{1-zq^{|J|}}.$$

Sums for the two row case

Sums for the two row case

Theorem (K.–Russell)

For $0 \leq b \leq \lfloor \frac{\ell}{2} \rfloor$, we have:

$$
\begin{aligned}
T_{(\ell-b,b)}(z,q) &= T_{(b,\ell-b)}(z,q) \\
&= \sum_{N_1,\cdots,N_\ell \geq 0} \frac{z^{N_1} q^{\sum_{i=1}^{\ell} \binom{N_i+1}{2} - \sum_{i=1}^{b} N_{2i}}}{(zq;q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}_q \begin{bmatrix} N_2 \\ N_3 \end{bmatrix}_q \cdots \begin{bmatrix} N_{\ell-1} \\ N_\ell \end{bmatrix}_q
\end{aligned}
$$

Proof sketch

Proof sketch

These types of sums satisfy their own (very easily derived) functional equations.

Proof sketch

These types of sums satisfy their own (very easily derived) functional equations.

Show that the required functional equations for tight cylindrics are consequences of, i.e., lie in a vector space (over $\mathbb{C}(z;q)$) spanned by the functional equations satisfied by the sums.

Proof sketch

These types of sums satisfy their own (very easily derived) functional equations.

Show that the required functional equations for tight cylindrics are consequences of, i.e., lie in a vector space (over $\mathbb{C}(z;q)$) spanned by the functional equations satisfied by the sums.

This is a linear algebra problem, can be solved easily on computer for low ℓ.

Proof sketch

These types of sums satisfy their own (very easily derived) functional equations.

Show that the required functional equations for tight cylindrics are consequences of, i.e., lie in a vector space (over $\mathbb{C}(z;q)$) spanned by the functional equations satisfied by the sums.

This is a linear algebra problem, can be solved easily on computer for low ℓ.

Identify patterns in the linear combinations for low ℓ, and hope that they generalize for all ℓ.

This technique has been used extensively in the last few years.

Open Question

Find/guess bivariate generating functions for 3-rowed tight cylindric partitions.

DHK partitions

DHK partitions

DHK partitions

(Dousse–Hardiman–Konan)

DHK partitions

(Dousse–Hardiman–Konan)

Let $\ell \geq 1$ and $0 \leq b \leq \ell$.

Let $\mathrm{DHK}_{b,\ell}$ be the set of $\ell+1$ colored partitions $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s = 0$, with allowed colors being $0, 1, \cdots, \ell$, satisfying:

DHK partitions

(Dousse–Hardiman–Konan)

Let $\ell \geq 1$ and $0 \leq b \leq \ell$.

Let $\mathrm{DHK}_{b,\ell}$ be the set of $\ell+1$ colored partitions $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s = 0$, with allowed colors being $0, 1, \cdots, \ell$, satisfying:

- $\lambda_i - \lambda_{i+1} = |u_i - u_{i+1}|$ for all i where u_i is the color of the part λ_i,

DHK partitions

(Dousse–Hardiman–Konan)

Let $\ell \geq 1$ and $0 \leq b \leq \ell$.

Let $\mathrm{DHK}_{b,\ell}$ be the set of $\ell+1$ colored partitions $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s = 0$, with allowed colors being $0, 1, \cdots, \ell$, satisfying:

- $\lambda_i - \lambda_{i+1} = |u_i - u_{i+1}|$ for all i where u_i is the color of the part λ_i,
- the color of $\lambda_s = 0$ is b,

DHK partitions

(Dousse–Hardiman–Konan)

Let $\ell \geq 1$ and $0 \leq b \leq \ell$.

Let $\mathrm{DHK}_{b,\ell}$ be the set of $\ell+1$ colored partitions $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s = 0$, with allowed colors being $0, 1, \cdots, \ell$, satisfying:

- $\lambda_i - \lambda_{i+1} = |u_i - u_{i+1}|$ for all i where u_i is the color of the part λ_i,
- the color of $\lambda_s = 0$ is b,
- there is exactly one part of size 0.

Example

Let $\pi = 8_2 + 7_3 + 7_3 + 5_1 + 4_0 + 3_1 + 3_1 + 3_1 + 2_0 + 1_1 + 0_0$ (here, subscripts are colors).

Then, $\pi \in \mathrm{DHK}_{0,3}$, $\#(\pi) = 10$, $\mathrm{wt}(\pi) = 43$.

We ignore the last 0 part while counting the number of parts,

The number of parts of $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s = 0$ is $s-1$.

The symbol $\#$ denotes the number of parts.

$$D_{b,\ell}(z,q) = \sum_{\lambda \in \mathrm{DHK}_{b,\ell}} z^{\#(\lambda)} q^{\mathrm{wt}(\lambda)}.$$

DHK identity

DHK identity

For $\ell \geq 1$, $0 \leq b \leq \ell$, we have:

$$D_{b,\ell}(1,q) = \frac{(q^{b+1}, q^{\ell-b+1}, q^{\ell+2}; q^{\ell+2})}{(q;q^2)_\infty (q;q)_\infty}.$$

DHK identity

For $\ell \geq 1$, $0 \leq b \leq \ell$, we have:

$$D_{b,\ell}(1,q) = \frac{(q^{b+1}, q^{\ell-b+1}, q^{\ell+2}; q^{\ell+2})}{(q;q^2)_\infty (q;q)_\infty}.$$

Recall: This is exactly the product $T_{(b,\ell-b)} = T_{(\ell-b,b)}$.

Question

How are DHK partitions and two-row tight cylindrics related?

Question

How are DHK partitions and two-row tight cylindrics related?

Answer

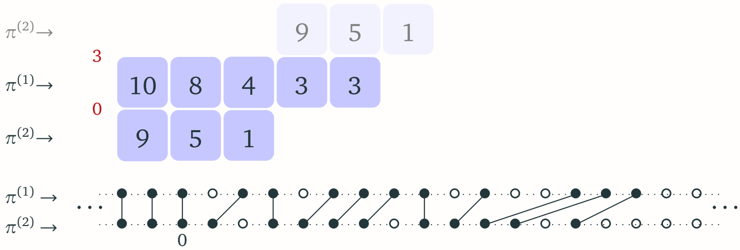
There is a very straight-forward bijection.

Bijection by example: Profile $(3, 0)$

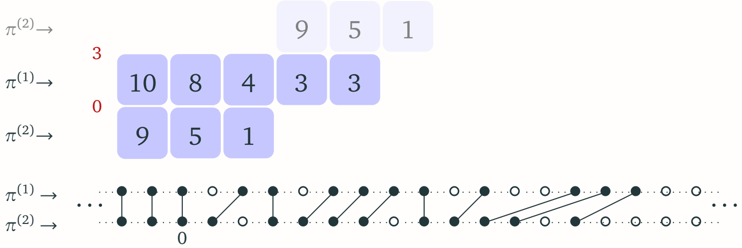
Bijection by example: Profile $(3, 0)$

$\pi^{(2)}\rightarrow$					9	5	1
	3						
$\pi^{(1)}\rightarrow$		10	8	4	3	3	
	0						
$\pi^{(2)}\rightarrow$		9	5	1			

Bijection by example: Profile $(3,0)$



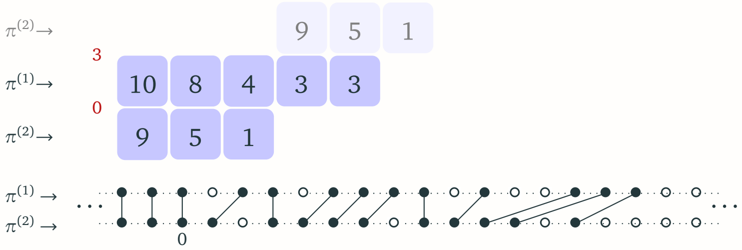
Bijection by example: Profile $(3, 0)$



Write down the total number of vacancies to the left of each yoke.

$0, 1, 2, 3, 3, 3, 4, 5, 7, 7, 8$

Bijection by example: Profile $(3, 0)$



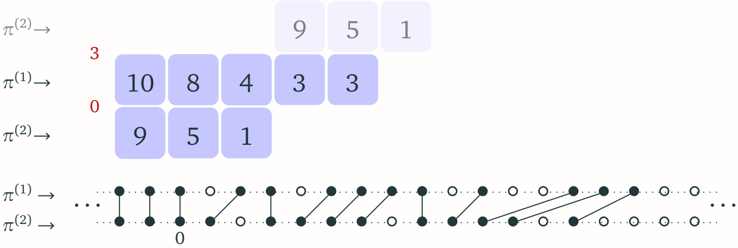
Write down the total number of vacancies to the left of each yoke.

$0, 1, 2, 3, 3, 3, 4, 5, 7, 7, 8$

Color with the “slant” of each yoke

$0_0 + 1_1 + 2_0 + 3_1 + 3_1 + 3_1 + 4_0 + 5_1 + 7_3 + 7_3 + 8_2.$

Bijection by example: Profile $(3, 0)$



Write down the total number of vacancies to the left of each yoke.

$0, 1, 2, 3, 3, 3, 4, 5, 7, 7, 8$

Color with the “slant” of each yoke

$0_0 + 1_1 + 2_0 + 3_1 + 3_1 + 3_1 + 4_0 + 5_1 + 7_3 + 7_3 + 8_2.$

This is in $\mathrm{DHK}_{0,3}$ when written in weakly decreasing order.

$(z;q)$ sums for DHK partitions

Theorem (K.–Russell)

For $0 \leq b \leq \lfloor \frac{\ell}{2} \rfloor$*, we have:*

$$\begin{aligned} D_{(b,\ell)}(z,q) &= D_{(\ell-b,\ell)}(z,q) \\ &= \sum_{N_1,\cdots,N_\ell \geq 0} \frac{z^{N_1} q^{\sum_{i=1}^{\ell} \binom{N_i+1}{2} - \sum_{i=1}^{b} N_{2i}}}{(zq;q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}_q \begin{bmatrix} N_2 \\ N_3 \end{bmatrix}_q \cdots \begin{bmatrix} N_{\ell-1} \\ N_\ell \end{bmatrix}_q \end{aligned}$$

Questions

Direct proof?

Our proof of bivariate sum-sides is indirect. It relies on uniqueness of solutions for a system of equations.

Is there a direct enumerative proof?

Kleshchev multipartitions

For a fixed profile c of rank 2, both DHK partitions and tight cylindric partitions have the structure of a crystal graph for a highest-weight integrable $\widehat{\mathfrak{sl}}_2$ module.

But, there are other combinatorial models for these crystal graphs.

Definition

Let $0 \leq b \leq \ell$.

A tuple of partitions $\pi = (\pi^{(1)}, \cdots, \pi^{(\ell)})$ is called a Kleshchev multipartition if:

- Each $\pi^{(i)}$ is a strict partition.
- $\pi_1^{(i)} \leq \#(\pi^{(i+1)})$ for all $i \neq b$ and $\pi_1^{(b)} \leq \#(\pi^{(b+1)}) + 1$.

$K_{(b,\ell)}(q)$: univariate generating function

Product form

For $0 \leq b \leq \ell$,

$$K_{(b,\ell)}(q) = \frac{(q^{b+1}, q^{\ell-b+1}, q^{\ell+2}; q^{\ell+2})_\infty}{(q;q^2)_\infty (q)_\infty}$$

Product form

For $0 \leq b \leq \ell$,

$$K_{(b,\ell)}(q) = \frac{(q^{b+1}, q^{\ell-b+1}, q^{\ell+2}; q^{\ell+2})_\infty}{(q;q^2)_\infty (q)_\infty}$$

This is the same as the univariate generating function $T_{(b,\ell-b)}(q) = T_{(\ell-b,b)}(q)$.

Product form

For $0 \leq b \leq \ell$,

$$K_{(b,\ell)}(q) = \frac{(q^{b+1}, q^{\ell-b+1}, q^{\ell+2}; q^{\ell+2})_\infty}{(q;q^2)_\infty (q)_\infty}$$

This is the same as the univariate generating function $T_{(b,\ell-b)}(q) = T_{(\ell-b,b)}(q)$.

Question

Is there a nice bijection between two-row tight cylindric partitions and Kleshchev multipartitions?

Parity considerations in Andrews–Gordon identities

Recall our sums, but set $z \mapsto 1$:

For $0 \leq b \leq \lfloor \frac{\ell}{2} \rfloor$, we have:

$$
\begin{aligned}
T_{(\ell-b,b)}(1,q) &= T_{(b,\ell-b)}(1,q) \\
&= \sum_{N_1,\cdots,N_\ell \geq 0} \frac{q^{\sum_{i=1}^{\ell} \binom{N_i+1}{2} - \sum_{i=1}^{b} N_{2i}}}{(q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}_q \begin{bmatrix} N_2 \\ N_3 \end{bmatrix}_q \cdots \begin{bmatrix} N_{\ell-1} \\ N_\ell \end{bmatrix}_q
\end{aligned}
$$

These sums have appeared previously in papers of Andrews, Kim–Yee, Kurşungöz with regards to parity considerations in Andrews–Gordon identities.

These are in turn related to Kleshchev multipartitions in the paper of Chern–Li–Stanton–Xue–Yee.

These sums have appeared previously in papers of Andrews, Kim–Yee, Kurşungöz with regards to parity considerations in Andrews–Gordon identities.

These are in turn related to Kleshchev multipartitions in the paper of Chern–Li–Stanton–Xue–Yee.

Question

Is there a relationship with tight cylindric partitions?

Unimodality

Unimodality

Fix a profile c, the coefficient of any fixed power of q is a polynomial in z.

Unimodality

Fix a profile c, the coefficient of any fixed power of q is a polynomial in z.

Experimentally, it appears that for both cylindric and tight cylindric partitions, the coefficients of these polynomials are unimodal, i.e., they increase first, reach the maximum at either one or two nearby coefficients and then decrease.

Unimodality

Fix a profile c, the coefficient of any fixed power of q is a polynomial in z.

Experimentally, it appears that for both cylindric and tight cylindric partitions, the coefficients of these polynomials are unimodal, i.e., they increase first, reach the maximum at either one or two nearby coefficients and then decrease.

Question

It would be nice to have a proof of this unimodality.

Example

$$
\begin{aligned}
&T_{(2,0,0)}(z;q)\\
&\qquad = 1 + zq + (z^2 + 2z)q^2 + (z^3 + 2z^2 + z)q^3 + (z^4 + 2z^3 + 4z^2 + z)q^4\\
&\qquad + (z^5 + 2z^4 + 4z^3 + 4z^2)q^5 + (z^6 + 2z^5 + 4z^4 + 6z^3 + 5z^2)q^6\\
&\qquad + (z^7 + 2z^6 + 4z^5 + 6z^4 + 9z^3 + 4z^2)q^7\\
&\qquad + (z^8 + 2z^7 + 4z^6 + 6z^5 + 11z^4 + 12z^3 + 4z^2)q^8\\
&\qquad + (z^9 + 2z^8 + 4z^7 + 6z^6 + 11z^5 + 15z^4 + 14z^3 + 2z^2)q^9\\
&\qquad + (z^{10} + 2z^9 + 4z^8 + 6z^7 + 11z^6 + 17z^5 + 22z^4 + 16z^3 + 2z^2)q^{10}\\
&\qquad + \cdots
\end{aligned}
$$

Thank you!