Combinatorial proofs and refinements of three partition theorems of Andrews

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Plan of the talk

(Restricted) Two-color partitions

Andrews' three theorems and our refinements

Proofs of refinements R1 and R2

Two papers dedicated to Srinivasa Ramanujan

Hardy-Ramanujan Journal 44 (2021), 74-80 submitted 18/06/2021, accepted 12/07/2021, revised 04/07/2021

Partition Identities for Two-Color Partitions

George E. Andrews*

In memory of Srinivasa Ramanujan

Abstract. Three new partition identities are found for two-color partitions. The first relates to ordinary partitions into parts not divisible by 4, the second to basis partitions, and the third to partitions with distinct parts. The surprise of the strangeness of this trio becomes clear in the proof.

Keywords. Partitions, Two-Color Partitions, Rogers-Ramanujan identities.

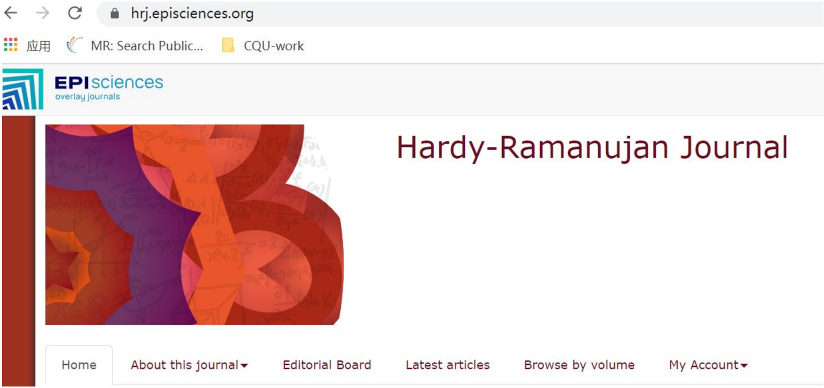
2010 Mathematics Subject Classification. 11P81, 05A19

1. Introduction.

The first paper on this topic [And87] appeared in 1987 to celebrate the $100^{\underline{\text{th}}}$ anniversary of Ramanujan's birth. It seems fitting to continue the study in this volume marking the $100^{\underline{\text{th}}}$ anniversary of his death.

G.E. Andrews, Rogers-Ramanujan identities for two-color partitions, Indian Journal of Mathematics, **29** (1987), 117–125.

Hardy-Ramanujan Journal with special volumes 43–44



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Let $n \geq 0$ and $d \geq 1$ be integers. Following Andrews, we define $\mathcal{L}_d(n)$ to be the set of two-color partitions of n into numerically distinct parts such that the following three conditions are satisfied:

1. each red part is at least d larger than the next largest part;
2. each green part is at least $d+1$ larger than the next largest part;
3. neither 1_g nor $(d-1)_g$ is allowed as a part.

We use $L_d(n)$ to denote the cardinality of $\mathcal{L}_d(n)$.

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We use $L_d(n)$ to denote the cardinality of $\mathcal{L}_d(n)$.

As remarked by Lovejoy, certain restricted two-color partitions are essentially **overpartitions**.

Two-color partitions in the literature (sampler)

- P. Hammond and R. Lewis, Congruences in ordered pairs of partitions, *Int. J. Math. Math. Sci.*, (2004).
- F.G. Garvan, Biranks for partitions into 2 colors, in *Ramanujan rediscovered*, *Ramanujan Math. Soc. Lect. Notes Ser.* 14, (Ramanujan Math. Soc., Mysore, 2010).
- W.Y.C. Chen and B.L.S. Lin, Congruences for bipartitions with odd parts distinct, *Ramanujan J.*, (2011).
- G.E. Andrews and P. Paule, MacMahon's pratition analysis XIII: Schmidt type partitions and modular forms, *J. Number Theory*, (2021).
- G.E. Andrews and W. Keith, A general class of Schmidt theorems, *J. Number Theory*, (2023).

Three theorems of Andrews

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Remark

To be precise, the original statements of Theorems L1 and L3 in Andrews' paper are different from what we give here, but it suffices to use Euler's celebrated "Odd-distinct Theorem" to bridge the gap. The definition of basis partition is a bit involved and will be given later.

Andrews' original proofs

To prove his three theorems on two-color partitions, Andrews bounded the largest part and utilized Abel's lemma to derive the generating functions for $L_1(n), L_2(n)$ and $L_3(n)$.

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Lemma

If $\lim_{n\to\infty} a_n = L$*, then*

$$\lim_{x\to 1^-}(1-x)\sum_{n\geq 0} a_n x^n = L.$$

Then he resorted to appropriate q-series identities (such as Lebesgue identity and Sylvester identity) and standard manipulations to make connections.

A refinement of Theorem L1

Definition

Let $\mathcal{L}_d(n,k,\ell)$ (resp. $L_d(n,k,\ell)$) denote the set (resp. the number) of two-color partitions in $\mathcal{L}_d(n)$ with k red parts and ℓ green parts.

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Let $\mathcal{A}(n,k,\ell)$ (resp. $A(n,k,\ell)$) be the set (resp. the number) of two-color partitions of n, each of which is consisted of $k+j$ distinct red parts and ℓ distinct even green parts for a certain non-negative integer j, wherein exactly k red parts are larger than ℓ.

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Theorem (R1)

For non-negative integers n, k, ℓ, we have $L_1(n,k,\ell) = A(n,k,\ell)$.

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Example

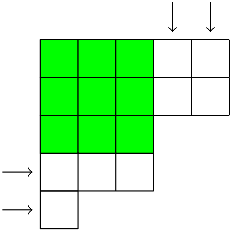
Taking $(n,k,\ell)=(7,1,1)$, we see that

$$\mathcal{L}_1(7,1,1)=\{6_g+1_r,\ 5_g+2_r,\ 5_r+2_g,\ 4_r+3_g\},$$

and

$$\mathcal{A}(7,1,1)=\{4_g+2_r+1_r,\ 4_r+2_g+1_r,\ 4_g+3_r,\ 5_r+2_g\}.$$

Durfee square and basis partition



$\longmapsto \quad (3, 3+1, 2+2)$

Fig.: Durfee square and triple notation of $\lambda = 5+5+3+3+1$.

G. E. Andrews, Basis partition polynomials, overpartitions and the Rogers-Ramanujan identities, J. Approx. Theory **197** (2015), 62–68.

H. Gupta, The rank-vector of a partition, Fibonacci Quart. **16** (1978), 548–552.

J. M. Nolan, C. D. Savage and H. S. Wilf, Basis partitions, Discrete Math. **179** (1998), 277–283.

A refinement of Theorem L2

Theorem (R2)

The number of basis partitions $\lambda = (k+\ell, \pi, \sigma)$ *of* n *such that* π *has exactly* ℓ *distinct parts, is given by* $L_2(n, k, \ell)$.

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Example

For $(n,k,\ell) = (15,1,2)$, *there are six basis partitions meeting the requirements:*

$$(3, 3+1+1+1, \epsilon), (3, 2+2+1+1, \epsilon), (3, 2+1+1+1+1, \epsilon),$$
$$(3, 3+2, 1), (3, 3+1, 2), (3, 2+1, 3).$$

On the other hand, one checks that

$$\mathcal{L}_2(15,1,2) = \{10_g+4_g+1_r, 9_g+5_g+1_r, 8_g+5_g+2_r, 9_g+4_r+2_g, 8_g+5_r+2_g, 8_r+5_g+2_g\}.$$

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$L_3(n, k, \ell)$ equals the number of partitions of n into $k + 2\ell$ distinct parts such that the Durfee square is of side $k + \ell$.

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Example

For $(n,k,\ell)=(18,2,1)$, the following are the ten strict partitions meeting the requirements,

$$10+4+3+1,\ 9+5+3+1,\ 8+6+3+1,\ 8+5+4+1,\ 7+6+4+1,$$
$$9+4+3+2,\ 8+5+3+2,\ 7+6+3+2,\ 7+5+4+2,\ 6+5+4+3.$$

On the other hand, one checks that $\mathcal{L}_3(18,2,1)$ contains ten partitions as well:

$$13_g+4_r+1_r,\ 12_g+5_r+1_r,\ 12_r+5_g+1_r,\ 11_g+6_r+1_r,\ 11_r+6_g+1_r,$$
$$10_r+7_g+1_r,\ 11_g+5_r+2_r,\ 10_g+6_r+2_r,\ 10_r+6_g+2_r,\ 9_r+6_r+3_g.$$

Analytic proof of R1

Recall the standard q-rising factorial notations:

$$(a)_n = (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}),$$
$$(a)_\infty = (a;q)_\infty = \lim_{n\to\infty}(a;q)_n, \text{ and } (a)_0 = 1.$$

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We claim the following two generating functions for $L_1(n,k,\ell)$ and $A(n,k,\ell)$, respectively.

$$\sum_{n,k,\ell\geq 0} L_1(n,k,\ell)x^k y^\ell q^n = \sum_{m\geq 0}\frac{(-yq/x)_m x^m q^{\binom{m+1}{2}}}{(q)_m}, \tag{1}$$

$$\sum_{n,k,\ell\geq 0} A(n,k,\ell)x^k y^\ell q^n = \sum_{k,\ell\geq 0}\frac{x^k y^\ell q^{\binom{k+\ell+1}{2}+\binom{\ell+1}{2}}}{(q)_k(q)_\ell}. \tag{2}$$

Proof of (2)

Recall that a partition in $\mathcal{A}(n, k, \ell)$ is consisted of $k + j$ distinct red parts and ℓ distinct even green parts for a certain non-negative integer j, wherein exactly k red parts are larger than ℓ.

Proof of (2)

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$$k \text{ red parts larger than } \ell\text{: } \frac{x^k q^{(\ell+1)+(\ell+2)+\cdots+(\ell+k)}}{(q)_k} = \frac{x^k q^{k\ell+\binom{k+1}{2}}}{(q)_k};$$

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So collectively we have

$$\sum_{n,k,\ell\geq 0} A(n,k,\ell)x^k y^\ell q^n = \sum_{k,\ell\geq 0} \frac{x^k y^\ell q^{\binom{k+\ell+1}{2}+\binom{\ell+1}{2}}}{(q)_k(q)_\ell}.$$

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q-binomial theorem: $\sum_{n=0}^{\infty} \frac{(a)_n t^n}{(q)_n} = \frac{(at)_\infty}{(t)_\infty}$, for $|q| < 1, |t| < 1$.

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$$\begin{aligned}
\sum_{k,\ell \geq 0} \frac{x^k y^\ell q^{\binom{k+\ell+1}{2} + \binom{\ell+1}{2}}}{(q)_k (q)_\ell} &= \sum_{m \geq 0} \frac{x^m q^{\binom{m+1}{2}}}{(q)_m} \sum_{\ell=0}^{m} \frac{(q^{m-\ell+1})_\ell}{(q)_\ell} \left(\frac{y}{x}\right)^\ell q^{\binom{\ell+1}{2}} \\
&= \sum_{m \geq 0} \frac{x^m q^{\binom{m+1}{2}}}{(q)_m} \sum_{\ell=0}^{m} \frac{(q^{-m})_\ell}{(q)_\ell} \left(-\frac{yq^{m+1}}{x}\right)^\ell \\
&= \sum_{m \geq 0} \frac{x^m q^{\binom{m+1}{2}}}{(q)_m} \frac{(-yq/x)_\infty}{(-yq^{m+1}/x)_\infty} \quad \text{(by } q\text{-binomial theorem)} \\
&= \sum_{m \geq 0} \frac{x^m q^{\binom{m+1}{2}}}{(q)_m} (-yq/x)_m,
\end{aligned}$$

which is the right hand side of (1), as desired.

Bijective proof of R1–I: Lebesgue identity

Andrews's original proof of Theorem L1 relies on the following identity of Lebesgue:

$$\sum_{n\geq 0} \frac{(-zq;q)_n}{(q;q)_n} q^{\binom{n+1}{2}} = (-q;q)_\infty(-zq^2;q^2)_\infty.$$

J. Dousse and B. Kim, An overpartition analogue of q-binomial coefficients, II: Combinatorial proofs and (q,t)-log concavity, J. Combin. Theory Ser. A **158** (2018), 228–253.

D. P. Little and J. A. Sellers, New proofs of identities of Lebesgue and Göllnitz via tilings, J. Combin. Theory Ser. A **116** (2009), 223–231.

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Bijective proof of R1–II: profile word

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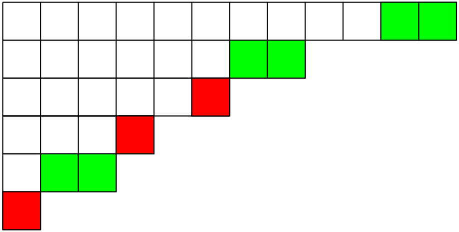

$$\stackrel{\phi}{\longmapsto} \quad x\ z\ x\ y\ x\ z\ y\ y\ z$$

Fig.: the profile of partition $\lambda = 12_g + 8_g + 6_r + 4_r + 3_g + 1_r$.

Bijective proof of R1–III: parameter j

Let $A = \{x, y, z\}$ be our alphabet, and A^* stands for the free monoid generated by the letters from A. We denote $\mathcal{W}$ the set of words in A^* that are either empty or end with x or z. For each word $u = u_1 u_2 \cdots u_m \in \mathcal{W}$, we define its *weight* to be

$$\omega(u) := \sum_{i=1}^{m} \chi(u_i \neq y) \cdot (i + |\{j \leq i : u_j = z\}|),$$

where $\chi(S) = 1$ if the statement S is true and $\chi(S) = 0$ otherwise. For the word $u = xzxyxzyyz$ in Fig. 2, one checks that $\omega(u) = 34$. We denote $\mathcal{W}(n, k, \ell)$ the set of words in A^* having weight n, wherein the letter x appears k times and the letter z appears ℓ times.

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Definition

For a word $u = u_1 u_2 \cdots u_m \in A^*$, we say a letter $u_i = z$ is odd (resp. even) if for all $1 \leq j < i$, there is an odd (resp. even) number of u_j being a y or an odd z. For each $0 \leq j \leq \ell$, we denote $\mathcal{W}(n, k, \ell, j)$ the set of words in $\mathcal{W}(n, k, \ell)$ that contain j odd z's.

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For the word $u = xzxyxzyyz$ in Fig. 2, from left to right, the first and the last z are even while the middle z is odd, hence $u \in \mathcal{W}(34, 3, 3, 1)$.

Bijective proof of R1–IV: the bijection ψ

Theorem (R1+)

There exists a bijection $\psi : \mathcal{W}(n, k, \ell, j) \to \mathcal{A}(n, k, \ell, j)$.

Here for $0 \leq j \leq \ell$, $\mathcal{A}(n, k, \ell, j)$ is the set of partitions in $\mathcal{A}(n, k, \ell)$ with precisely j red parts no greater than ℓ.

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证明.

Take any word $u = u_1 u_2 \cdots u_m \in \mathcal{W}(n,k,\ell,j)$, we aim to derive a partition pair $\psi(u) := (\pi, \sigma)$, where π is a distinct partition into $k+j$ parts while σ is a distinct partition into ℓ even parts.

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Step 1 Screen the word u R-to-L looking for letter z, and suppose they are $u_{s_1}, u_{s_2}, \ldots, u_{s_\ell}$ in u with $s_1 > s_2 > \cdots > s_\ell$.

Step 2 For $i = 1, 2, \ldots, \ell$, we let

$$\sigma_i := s_i - |\{t \le s_i : u_t = x\}| + |\{t \le s_i : u_t \text{ is an even } z\}|,$$

$$v_i := \begin{cases} x, & \text{if } u_{s_i} \text{ is an odd } z, \\ y, & \text{if } u_{s_i} \text{ is an even } z, \end{cases}$$

$$\hat{u} := v_1 v_2 \cdots v_\ell u', \quad u'(z \to y),$$

set $\pi := \phi^{-1}(\hat{u})$ and $\sigma := \sigma_1 + \sigma_2 + \cdots + \sigma_\ell$.

Bijective proof of R1–V: the inverse mapping ψ^{-1}

Example

Given a two-color partition $\lambda \in \mathcal{A}(50, 3, 4, 2)$*,*

$$\lambda = 10_r + 10_g + 8_g + 6_r + 5_r + 4_r + 4_g + 2_g + 1_r = (10+6+5+4+1)_r + (10+8+4+2)_g,$$

we show how to recover its preimage $\psi^{-1}(\lambda) \in \mathcal{W}(50, 3, 4, 2)$*.*

Bijective proof of R1–V: the inverse mapping ψ^{-1}

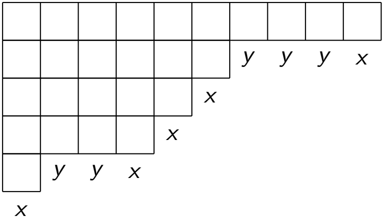
Example

Given a two-color partition $\lambda \in \mathcal{A}(50,3,4,2)$,

$$\lambda = 10_r+10_g+8_g+6_r+5_r+4_r+4_g+2_g+1_r = (10+6+5+4+1)_r+(10+8+4+2)_g,$$

we show how to recover its preimage $\psi^{-1}(\lambda) \in \mathcal{W}(50,3,4,2)$.

$\pi =$



$$\xrightarrow{\phi} \qquad \hat{u} = \underbrace{xyyx}_{v}\,\underbrace{xxyyyxyy\ldots}_{u'}$$

Fig.: *the profile word of* $\pi = 10+6+5+4+1$, *with infinitely many* y*'s appended.*

$$
\begin{array}{ccccccccccccccc}
u' & = & x & x & y & y & y & x & y & y & y & y & y & y & \dots \\
 & & & & & \downarrow & & & & & & & & & \\
 & & x & x & y & z & y & x & y & y & y & y & y & y & \dots \\
 & & & & & & \downarrow & & & & & & & & \\
 & & x & x & y & z & z & x & y & y & y & y & y & y & \dots \\
 & & & & & & & & & & \downarrow & & & & \\
 & & x & x & y & z & z & x & y & y & z & y & y & y & \dots \\
 & & & & & & & & & & & & \downarrow & & \\
 & & x & x & y & z & z & x & y & y & z & y & z & = & u
\end{array}
$$

One checks that $u \in \mathcal{W}(50, 3, 4, 2)$ indeed.

Analytic counterpart?

Question

Recall our first proof of Theorem R1, where we have derived the generating function (2) *for* $A(n,k,\ell)$. *The same analysis readily gives us:*

$$\sum_{n,k,\ell,j\geq 0} A(n,k,\ell,j)x^k y^\ell z^j q^n = \sum_{k,\ell\geq 0} \frac{(-zq;q)_\ell}{(q;q)_k(q^2;q^2)_\ell} x^k y^\ell q^{\binom{k+\ell+1}{2}+\binom{\ell+1}{2}}.$$

Therefore, it might be interesting to find the generating function for $|\mathcal{W}(n,k,\ell,j)|$, *so as to give an analytic counterpart of Theorem R1+.*

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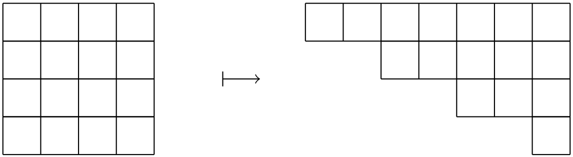

Fig.: $n^2 = 1 + 3 + \cdots + (2n-1)$.

A curious conjecture

Directly from definitions, we see that

$$L_1(n) \geq L_2(n) \geq L_3(n).$$

Let $B(n)$ be the number of basis partitions of n, $p_k(n)$ be the number of k-regular partitions of n, i.e., partitions in which no part is divisible by k.

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Conjecture (Andrews, 2021)

$p_3(n) \geq B(n)$ with strict inequality if $n > 3$.

Thank you all for listening!

talk based on my paper:
Combinatorial proofs and refinements of three partition theorems of Andrews, Ramanujan J, (2023).