Exceptional Congruences for Eta-Quotient Newforms

(Joint work with Henry Stone, Eddie O'Sullivan and Xiaolan Jin)

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Seminar in Partition Theory, q-Series and Related Topics
Michigan Technological University
arXiv:2511.16039

Motivation

Let $z \in \mathbb{H}$ and $q = e^{2\pi iz}$. We define

$$\eta(z) = q^{1/24} \prod_{n \geq 1} (1 - q^n) \qquad \text{(a weight 1/2 cusp form with eta-multiplier)}$$

$$\Delta(z) = \eta(z)^{24} \in S_{12}(\Gamma_0(1)),$$

$$= \sum_{n \geq 1} \tau(n) q^n \;=\; q - 24q^2 + 252q^3 - 1472q^4 + \cdots$$

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Serre-Swinnerton-Dyer (1973)
Congruences for normalized Hecke eigenforms in $S_k(\Gamma_0(1)) \cap \mathbb{Z}[[q]]$.
(Key insight) Use of modular Galois representations (Deligne-Serre).

Modular Galois representations

Theorem (Deligne)

Let $k \geq 2$ be an integer and $f = \sum_{n\geq 1} a(n)q^n \in S_k(\Gamma_0(N), \chi)$ be a normalized eigenform with ℓ-adic integer coefficients. Then for each prime ℓ, there exists a continuous homomorphism

$$\rho_{\ell,f} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\mathbb{Z}_\ell)$$

such that for every prime $p \nmid N\ell$, the characteristic polynomial of $\rho_{\ell,f}(\mathrm{Frob}_p)$ is

$$X^2 - a(p)X + \chi(p)\,p^{k-1}.$$

Write $\bar{\rho}_{\ell,f} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_\ell)$, for the reduction of $\rho_{\ell,f}$ modulo ℓ.

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Exceptional primes Let $\ell > 3$ be a prime. We say that ℓ is *exceptional* for f iff

$$\mathrm{Im}\left(\bar{\rho}_{\ell,f}\right) \text{ does not contain } \mathrm{SL}_2(\mathbb{F}_\ell).$$

Classification of exceptional primes

Let ℓ be an exceptional prime for a level-1 eigenform f with nebentypus χ. Let $G = \mathrm{Im}\,\bar{\rho}_{f,\ell} \subseteq \mathrm{GL}_2(\mathbb{F}_\ell)$ and let H be its projective image.

1. If G is contained in a Borel subgroup, then ℓ is Type I. For all $n \geq 1$ with $(n, \ell) = 1$,

$$a(n) \equiv n^m \sigma_{k-1-2m}(n) \pmod{\ell}.$$

Examples: $\tau(n) \equiv n\sigma_1(n) \pmod{5}$, $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$.

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Examples: $\tau(n) \equiv n\sigma_1(n) \pmod 5$, $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$.

2. If H is dihedral, then ℓ is Type II. For primes $n \nmid \ell$ with $\left(\frac{n}{\ell}\right) = -1$,
$$a(n) \equiv 0 \pmod{\ell}.$$
Example: If $\left(\frac{n}{23}\right) = -1$, then $\tau(n) \equiv 0 \pmod{23}$.

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Let ℓ be an exceptional prime for a level-1 eigenform f with nebentypus χ. Let $G = \mathrm{Im}\,\bar{\rho}_{f,\ell} \subseteq \mathrm{GL}_2(\mathbb{F}_\ell)$ and let H be its projective image.

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3. If $H \cong A_4, S_4$, or A_5, then ℓ is Type III. If $H \cong S_4$, for primes $p \nmid N\ell$,
$$\frac{a(p)^2}{\chi(p)\, p^{k-1}} \equiv 0, 1, 2, 4 \pmod{\ell}.$$
Example: For $f = E_4\Delta$, $\ell = 59$ is Type III.

Known results

Ribet (1975, 1985)
Extended Swinnerton-Dyer results to newforms of arbitrary weight k and level $N > 1$.
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Goal:

- Classify and prove Type I and Type II congruences for eta-quotient newforms modulo a prime.
- Extensions to prime powers.

Framework for Type I congruences

Let $f = \sum a(n)q^n \in S_k(\Gamma_0(N), \chi)$ be an eta-quotient newform.

The form f satisfies Type I congruence provided $\exists$ real-valued ψ, ϕ mod N and $m' \geq m$ s.t.

$$a(p) \equiv \psi(p)p^m + \phi(p)p^{m'} \pmod{\ell}, \quad \psi(p)\phi(p)p^{m'+m} \equiv \chi(p)p^{k-1} \pmod{\ell}.$$

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Theorem (Sullivan, Stone, S., Jin)

Let f and ψ be as defined above. Then

1. *If $3 \leq m' - m + 1 \leq \ell - 2$, then we have*

$$\theta(f \otimes 1_N) \equiv \theta^{m+1}(G_{m'-m+1} \otimes \psi 1_N) \pmod{\ell}.$$

Moreover, we have $\ell < k$ or $\ell \mid (a(p) - \psi(p)\,\sigma_{k-1}(p))$ for all primes $p \nmid N$.

2. *If $m' - m + 1 \in \{2, \ell - 1\}$, then we have*

$$\theta(f \otimes 1_N) \equiv \theta^{m+1}(G_{\ell+1} \otimes \psi 1_N) \pmod{\ell}.$$

Further, we have $\ell < k$.

List of Type I congruences

Values of parameters in Theorem 2

$f(z)$	Type I				
	ℓ	m	m'	ψ	(k, N)
$\Delta(z)$	3 5 7 691	0 1 1 0	1 2 4 11	1_1	$(12, 1)$
$\eta(z)^8\eta(2z)^8$	2 3 5 17	0 0 1 0	1 1 2 7	1_2	$(8, 2)$
$\eta(z)^6\eta(3z)^6$	2 3 13	0 0 0	1 1 5	1_3 $\left(\frac{\cdot}{3}\right)$ 1_3	$(6, 3)$
$\eta(2z)^{12}$	2 3	0 0	1 1	1_4 1_4	$(6, 4)$
$\eta(z)^4\eta(5z)^4$	2 5 13	0 0 0	1 3 3	1_5 $\left(\frac{\cdot}{5}\right)$ 1_5	$(4, 5)$

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$f(z)$	Type I ℓ	m	m'	ψ	(k, N)
$\Delta(z)$	3	0	1	1_1	$(12, 1)$
	5	1	2		
	7	1	4		
	691	0	11		
$\eta(z)^8\eta(2z)^8$	2	0	1	1_2	$(8, 2)$
	3	0	1		
	5	1	2		
	17	0	7		
$\eta(z)^6\eta(3z)^6$	2	0	1	1_3	$(6, 3)$
	3	0	1	$\left(\frac{\cdot}{3}\right)$	
	13	0	5	1_3	
$\eta(2z)^{12}$	2	0	1	1_4	$(6, 4)$
	3	0	1	1_4	
$\eta(z)^4\eta(5z)^4$	2	0	1	1_5	$(4, 5)$
	5	0	3	$\left(\frac{\cdot}{5}\right)$	
	13	0	3	1_5	

$f(z)$	ℓ	m	m'	ψ	(k, N)
$\eta(z)^2\eta(2z)^2\eta(3z)^2\eta(6z)^2$	2	0	1	1_6	$(4, 6)$
	3	0	1	$1_2\left(\frac{\cdot}{3}\right)$	
	5	0	3	1_6	
$\eta(3z)^8$	2	0	1	1_9	$(4, 9)$
	3	0	1	$1_3\left(\frac{\cdot}{3}\right)$	
$\eta(z)^2\eta(11z)^2$	5	0	1	1_{11}	$(2, 11)$
$\eta(z)\eta(2z)\eta(7z)\eta(14z)$	2	0	1	1_{14}	$(2, 14)$
	3	0	1		
$\eta(z)\eta(3z)\eta(5z)\eta(15z)$	2	0	1	1_{15}	$(2, 15)$
$\eta(2z)^2\eta(10z)^2$	2	0	1	1_{20}	$(2, 20)$
	3	0	1		
$\eta(3z)^2\eta(9z)^2$	3	0	1	$1_9\left(\frac{\cdot}{3}\right)$	$(2, 27)$
$\eta(6z)^4$	2	0	1	1_{36}	$(2, 36)$
	3	0	1	$1_{12}\left(\frac{\cdot}{3}\right)$	

Remarks on Type I congruences

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3. We can associate the elliptic curve $E : y^2 - y = x^3 - x$, to the eta-quotient newform $f(z) = \eta(z)^2\eta(11z)^2$, whose coefficients obey the congruence

$$\begin{aligned} a_E(p) &\equiv p + 1 \pmod{5} \text{ for } p \nmid 55 \\ p + 1 - \#E(\mathbb{F}_p) &\equiv p + 1 \pmod{5} \\ \#E(\mathbb{F}_p) &\equiv 0 \pmod{5}. \end{aligned}$$

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4. (Combinatorial Interpretation) Let

$$\sum_{n\geq 0} v(n)q^n = \prod_{n\geq 1}(1-q^n)^2(1-q^{11n})^2.$$

Then we have $v(n) = v_e(n) - v_o(n)$. Hence, we obtain for all $(n, 11) = 1$,

$$v(n-1) \equiv a_{11}(n) \pmod{5} \text{ where } a_{11}(n) = \sum_{\substack{d|n \\ \gcd(d,11)=1}} d.$$

Framework for Type II congruences

The form f satisfies a Type II congruence provided for all primes p with $\left(\frac{p}{\ell}\right) = -1$,

$$a(p) \equiv 0 \pmod{\ell}.$$

Theorem (Sullivan, Stone, S., Jin)

Let $f \in S_k(\Gamma_0(N), \chi)$ be an eta-quotient newform and let ℓ be an exceptional prime of Type II for f. Then we have

$$\Theta^{\frac{\ell+1}{2}}(f \otimes 1_N) \equiv \Theta(f \otimes 1_N) \pmod{\ell},$$

and we have $\ell < k$ or

$$\ell = \begin{cases} 2k-1 & \text{if} \quad f \mid U_\ell \not\equiv 0 \pmod{\ell} \\ 2k-3 & \text{if} \quad f \mid U_\ell \equiv 0 \pmod{\ell} \end{cases}.$$

List of Type II congruences

$f(z)$	ℓ	(k, N)	χ
$\Delta(z)$	23	$(12, 1)$	1_1
$\eta(z)^8\eta(2z)^8$	3	$(8, 2)$	1_2
$\eta(z)^6\eta(3z)^6$	3	$(6, 3)$	1_3
$\eta(2z)^{12}$ $\eta(2z)^{12} \otimes \left(\frac{-4}{\cdot}\right)$	3 11	$(6, 4)$ $(6, 16)$	1_2 1_2
$\eta(z)^4\eta(2z)^2\eta(4z)^4$ $\eta(z)^4\eta(2z)^2\eta(4z)^4 \otimes \left(\frac{-8}{\cdot}\right)$	7	$(5, 4)$ $(5, 64)$	$\left(\frac{-4}{\cdot}\right)$ $\left(\frac{-4}{\cdot}\right)$
$\eta(z)^3\eta(7z)^3$	3	$(3, 7)$	$\left(\frac{-7}{\cdot}\right)$
$\eta(3z)^8$	3 5 7	$(4, 9)$	1_3
$\eta(2z)^3\eta(6z)^3$ $\eta(2z)^3\eta(6z)^3 \otimes \left(\frac{-4}{\cdot}\right)$	3	$(3, 12)$ $(3, 48)$	$\left(\frac{-3}{\cdot}\right)$ $\left(\frac{-3}{\cdot}\right)$
$\eta(z)\eta(2z)\eta(7z)\eta(14z)$	3	$(2, 14)$	1_{14}
$\eta(4z)^6$ $\frac{\eta(8z)^{18}}{\eta(4z)^6\eta(16z)^6}$ $= \eta(4z)^6 \otimes \left(\frac{-8}{\cdot}\right)$	3	$(3, 16)$ $(3, 64)$	$\left(\frac{-4}{\cdot}\right)$ $\left(\frac{-4}{\cdot}\right)$
$\eta(2z)^2\eta(10z)^2$	3	$(2, 20)$	1_{20}
$\eta(6z)^4$ $\frac{\eta(12z)^{12}}{\eta(6z)^4\eta(24z)^4}$ $= \eta(6z)^4 \otimes \left(\frac{-4}{\cdot}\right)$	3	$(2, 36)$ $(2, 144)$	1_6 1_{12}

Remarks on Type II Congruences

1. We recall that a form f has CM by $\chi = \left(\frac{D_K}{\cdot}\right)$ associated to K if and only if $f \otimes \chi = f$. Serre's (1985) work implies

$$f \text{ is a CM newform } \implies \lim_{X\to\infty} \frac{\#\{0 \leq n \leq X \ : \ a(n) \neq 0\}}{X} = 0$$

When ℓ is exceptional of Type II for one of CM forms, its coefficients vanish modulo ℓ for a set of Dirichlet density $> 1/2$. We further remark that if f is a newform with odd weight and real coefficients, then Corollary 1.2 of Schutt (2009) asserts that f has quadratic nebentypus character ψ and it has CM by this character.

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2. Suppose that ℓ is exceptional of Type I with $m' - m = \frac{\ell-1}{2}$. Then we have, for all primes $p \nmid N\ell$,

$$a(p) \equiv p^m(1+p^{m'-m}) \equiv p^m\left(1 + p^{\frac{\ell-1}{2}}\right) \equiv p^m\left(1 + \left(\frac{p}{\ell}\right)\right) \equiv \begin{cases} 2p^m, & \left(\frac{p}{\ell}\right) = 1 \\ 0, & \left(\frac{p}{\ell}\right) = -1 \end{cases}$$

Hence, ℓ is exceptional of Type II.

Extensions of Type I to prime powers

Swinnerton-Dyer extends his congruence results on $N = 1$ to prime power modulus.

$$\tau(n) \equiv 1537\,\sigma_{11}(n) \pmod{2^{12}} \text{ if } n \equiv 5 \pmod 8$$

$$\tau(n) \equiv n^{-30}\,\sigma_{71}(n) \pmod{5^3} \text{ if } \gcd(n,5) = 1.$$

$$\tau(n) \equiv n^{-610}\,\sigma_{1231}(n) \begin{cases} \pmod{3^6} \text{ if } n \equiv 1 \pmod 3, \\ \pmod{3^7} \text{ if } n \equiv 2 \pmod 3. \end{cases}$$

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Our next result is a $N > 1$ analogue of these congruences.

Theorem (Sullivan, Stone, S., Jin)

Let ℓ be an exceptional prime of Type I for an eta-quotient newform f. Let $t > 1$ and let $0 \le m < m' \le \phi(\ell^t)$ with $m + m' \equiv k - 1 \pmod{\phi(\ell^t)}$. The following table gives congruences of the form,

$$a(p) \equiv p^m + p^{m'} \pmod{\ell^t} \text{ for all } p \equiv b \pmod d. \tag{1}$$

Parameter values in Theorem 4

$f(z)$	ℓ	(m,m')	t	b	d
$\eta(z)^8\eta(2z)^8$	2	(0,7)	6	all residue classes	64
	3	(12,13)	3	all residue classes	27
$\eta(z)^6\eta(3z)^6$	2	(0,5)	4	5, 7, 11, 19	24
	2	(0,5)	5	13, 17, 23	24
	2	(0,5)	6	1	24
$\eta(2z)^{12}$	2	(0,5)	8	3	8
	2	(0,5)	9	7	8
	2	(0,5)	10	5	8
	2	(0,5)	11	1	8
	3	(1,4)	2	2, 5	9
	3	(1,4)	3	8, 17, 26	27
$\eta(z)^4\eta(5z)^4$	5	(1,2)	2	1, 6, 7, 11, 16, 18, 21, 24	25
$\eta(3z)^8$	2	(0,1)	2	3	4
	3	(0,3)	4	$\{3n+1 : 0 \leq n \leq 26\} \cup \{26, 53, 80\}$	81
$\eta(z)^2\eta(2z)^2\eta(3z)^2\eta(6z)^2$	2	(0,1)	2	all residue classes	4
$\eta(z)^2\eta(11z)^2$	5	(0,1)	2	1, 6, 11, 16, 21	25
$\eta(z)\eta(2z)\eta(7z)\eta(14z)$	3	(0,1)	2	1, 4, 7	9
$\eta(z)\eta(3z)\eta(5z)\eta(15z)$	2	(0,1)	3	all residue classes	8
$\eta(3z)^2\eta(9z)^2$	3	(0,1)	3	1, 10, 19, 26	27

Extensions of Type II to prime powers

Theorem (Sullivan, Stone, S., Jin)

Let ℓ be an exceptional prime of Type II for an eta-quotient newform f. Let t be a positive integer. The following table gives all congruences of the form

$$f \otimes 1_\ell \equiv f \otimes \left(\frac{\cdot}{\ell}\right) \pmod{\ell^t}.$$

$f(z)$	ℓ	t
$\eta(2z)^{12}$	3	1, 2, 3
$\eta(3z)^8$	3	≥ 1
$\eta(2z)^3\eta(6z)^3$	3	≥ 1
$\eta(3z)^2\eta(9z)^2$	3	≥ 1
$\eta(6z)^4$	3	≥ 1

Modular forms mod ℓ

- Let $\ell \geq 5$ be prime, and let $N \geq 1$. For $k \geq 0$,

$$M_k(\Gamma_0(N), \chi)_{(\ell)} := \{ f \in M_k(\Gamma_0(N), \chi) \; : \; q\text{-exp}(f) \in \mathbb{Z}_{(\ell)}[[q]] \}.$$

- We define $\widetilde{M}^{(\ell)}(\Gamma_0(N)) := \{ \widetilde{f} : f \in M_k(\Gamma_0(N), \chi)_{(\ell)} \} \subset \mathbb{F}_\ell[[q]]$.

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- We define $\widetilde{M}^{(\ell)}(\Gamma_0(N)) := \{ \widetilde{f} : f \in M_k(\Gamma_0(N), \chi)_{(\ell)} \} \subset \mathbb{F}_\ell[[q]]$.
- For $\widetilde{f} \neq 0$,

$$w_\ell(\widetilde{f}) := \min\left\{ k' \geq 0 : \exists\, g \in M_{k'}(\Gamma_0(N), \chi)_{(\ell)} \text{ with } \widetilde{g} = \widetilde{f} \right\}, \qquad w_\ell(0) := -\infty.$$

Modular forms mod ℓ

- Let $\ell \geq 5$ be prime, and let $N \geq 1$. For $k \geq 0$,

$$M_k(\Gamma_0(N), \chi)_{(\ell)} := \{\, f \in M_k(\Gamma_0(N), \chi) \;:\; q\text{-exp}(f) \in \mathbb{Z}_{(\ell)}[[q]] \,\}.$$

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- If $0 \neq f \in M_k(\Gamma_0(N), \chi)_{(\ell)}$, then

$$w_\ell(\widetilde{f}) \equiv k \pmod{\ell - 1} \quad \text{since} \quad \widetilde{E}_{\ell-1} = 1.$$

- For $t > 1$, the filtration modulo ℓ^t is well defined up to $(\ell - 1)\ell^{t-1}$.

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- Buzzard showed that Sturm Bound for forms with non-trivial Nebentypus agrees with Sturm bound for $\Gamma_0(N)$.

Theorem

Let $N \geq 1$, let ℓ be prime and let $f(z), g(z) \in M_k(\Gamma_0(N), \chi)_{(\ell)}$. Assume that

$$\mathrm{ord}_\ell(f - g) > \frac{k}{12}[\Gamma_0(1) : \Gamma_0(N)] \text{ where } [\Gamma_0(1) : \Gamma_0(N)] = N \prod_{p \text{ prime} : p|N} \left(1 + \frac{1}{p}\right).$$

Then we have $\mathrm{ord}_\ell(f(z) - g(z)) = \infty$.

The same bound works modulo ℓ^t for $t > 1$.

Ramanujan's Theta operator mod ℓ

- Ramanujan Theta operator

$$\theta := q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{dz}, \qquad \theta\left(\sum_{n\geq 0} a(n)q^n\right) = \sum_{n\geq 0} n\,a(n)\,q^n.$$

- Although θ does not preserve modularity, but it behaves nicely mod ℓ.

$$\theta : \ \widetilde{M}_k^{(\ell)}(\Gamma_0(1)) \longrightarrow \widetilde{M}_{k+\ell+1}^{(\ell)}(\Gamma_0(1)) \text{ for } \ell \geq 5.$$

- Filtration behavior: $w_\ell(\theta\widetilde{f}) = w_\ell(\widetilde{f}) + \ell + 1 - \alpha(\ell - 1), \quad \alpha = 0 \text{ if } \ell \nmid w_\ell(\widetilde{f})$.
- (Chen-Kiming) For $\ell \geq 5, t > 1$ and $\ell \nmid N$, the theta operator induces

$$\theta : \ \widetilde{M}_k^{(\ell^t)}(\Gamma_1(N)) \longrightarrow \widetilde{M}_{k+k(t)}^{(\ell^t)}(\Gamma_1(N)), \qquad k(t) = 2 + 2\,\ell^{t-1}(\ell - 1).$$

- (Katz-Gross) Extended these constructions to levels $N \geq 4$.

Theta Operator modulo powers of 2 and 3

Theorem (Sullivan, Stone, S., Jin)

Let $N \geq 1$ *and let* $\ell \in \{2, 3\}$. *We define*

$$j = j_{\ell,t} = \begin{cases} 2 + \phi(2^t), & \text{if } \ell = 2 \text{ and } t \geq 4 \\ 2 + \phi(3^t), & \text{if } \ell = 3 \text{ and } t \geq 2. \end{cases}$$

Then there exists an Eisenstein series $F_{\ell,t}(z) \in M_j\big(\Gamma_0(\ell^{t-1})\big)$ *such that*

$$E_2(z) \equiv F_{\ell,t}(z) \pmod{\ell^t}.$$

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Consequently, for every $f \in M_k(\Gamma_0(N), \chi)$ one has the congruence

$$\theta f = \vartheta f - \frac{k}{12} E_2 f \equiv \vartheta f - \frac{k}{12} F_{\ell,t} f \pmod{\ell^t}$$

Hence, we have $\theta f \in M_{j+k}(\Gamma_0(\ell^{t-1}N), \chi) \pmod{\ell^t}$.

Sketch of Proof of Theorem 2

- Let $N \geq 1$, and let $k \geq 2$ be even. We recall that f satisfies a Type I congruence if and only if the following holds:

$$a(p) \equiv \psi(p)p^m + \phi(p)p^{m'} \pmod{\ell},\ \psi(p)\phi(p)p^{m'+m} \equiv \chi(p)p^{k-1} \pmod{\ell}. \quad (2)$$

- Let ψ, ϕ be real-valued char. mod N and let $\chi = 1_N$. Let $\ell \geq 3$ be prime and $\ell \nmid N$. By Dirichlet's Theorem on primes in AP in (2), it follows that

$$m + m' \equiv k - 1 \pmod{\ell - 1}, \quad \psi\phi = 1_N \quad \text{and} \quad m \neq m'. \qquad (3)$$

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- We prove the result in case of $3 \leq m' - m + 1 \leq \ell - 2$.
- We note that since $f \in S_k(\Gamma_0(N), \chi)$ is a normalized Hecke eigenform, we have

$$a(m)a(n) = \sum_{d \mid \gcd(m,n)} \chi(d) d^{k-1} a\left(\frac{mn}{d^2}\right). \quad (4)$$

Proof (Continued)

- Let $E = G_{m'-m+1} \otimes \psi 1_N$. For all primes $p \nmid N\ell$, using (2) and (3), we have

$$a(p) \equiv p^m\, \psi(p)(1 + p^{m'-m}) \equiv p^m\, \psi(p)\, \sigma_{m'-m}(p) \pmod{\ell}.$$

- Multiplying both sides of by $p1_N(p)$, we have

$$p\, 1_N(p)\, a(p) \equiv p^{m+1}\, 1_N(p)\, \psi(p)\, \sigma_{m'-m}(p) \pmod{\ell}\ \forall\ p.$$

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- Since f and E are normalized Hecke eigenforms for all T_p, using multiplicativity (4) and applying induction on n gives:

$$n\, 1_N(n)\, a(n) \equiv n^{m+1}\, 1_N(n)\, \psi(n)\, \sigma_{m'-m}(n) \pmod{\ell}.$$

- We recall that the Eisenstein series has q-expansion

$$G_{m'-m+1}(z) = 1 - \frac{2(m'-m+1)}{B_{m'-m+1}} \sigma_{m'-m}(n) q^n \in M_{m'-m+1}(\Gamma_0(1)).$$

- Therefore, we have

$$\theta(f \otimes 1_N) \equiv \theta^{m+1}(E) \pmod{\ell}.$$

Bound on ℓ

- By induction on j, it follows that for all $0 \leq j \leq m$, we have

$$w_\ell(\theta^j E) = w_\ell(E) + j(\ell+1) \not\equiv 0 \pmod{\ell}.$$

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- Thus, we deduce that $m' + m\ell + 1 \le k$. If $m \neq 0$ then $m' > m \ge 1$ implies that $\ell \le k - 3 < k$.
- For $m = 0$, we have $m' = k - 1$, and one can show that either $\ell < k$ or the following holds:

$$f \otimes 1_N \equiv G_k \otimes \psi 1_N \pmod{\ell}$$

  implying that $\ell \mid (a(p) - \psi(p)\sigma_{k-1}(p))$.

Sketch of Proof of Theorem 7

- Let $\ell = 3$, let $t \geq 2$, let $j = 2 + \phi(3^t)$ and let $j' = 2$. Since $j \equiv j' \pmod{\phi(3^t)}$, by Kummer's congruence, we have

$$(\ell^{j-1} - 1)\,\frac{B_j}{j} + \frac{1 - \frac{1}{\ell}}{j} - j \equiv B_{j'} + \frac{1 - \frac{1}{\ell}}{j'} - j \pmod{\ell^t}. \tag{5}$$

- Using Clausen-Von Staudt Theorem, we deduce that

$$v_\ell\left(\ell^{j-1}\,\frac{B_j}{j}\right) = \phi(\ell^t) \geq t. \tag{6}$$

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- Hence, we conclude that

$$E_{2+\phi(3^t)}(z) = E_2^{1_1,1_3}(z) = -\frac{1}{2}E_2(z)\big|(1 - 3V_3) \pmod{3^t}. \tag{7}$$

- Acting with the operator $(1 - 3V_3)^{-1}$ on both sides of (7) gives the desired result.

Example

Let $f(z) = \sum a(n)q^n = \eta(z)^8\eta(2z)^8 \in S_8(\Gamma_0(2))$, and let $(\ell, t) = (3, 3)$.

- From Table 16, we have

$$a(p) \equiv p^{12} + p^{13} \equiv p^{12}(1+p) \equiv p^{12}(1+p^{19}) \pmod{27} \; \forall \quad \text{primes } p \notin \{2, 3\}.$$

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- Multiplying both sides by $p^3\, 1_2(p)$, we obtain

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- Using Theorem 7, we obtain that $\theta^3(f \otimes 1_2) \in S_{68}(\Gamma_0(36))$ and $\theta^{15}(G_{20} \otimes 1_2) \in S_{320}(\Gamma_0(36))$. Since $E_{18} \equiv 1 \pmod{27}$, to prove that

$$\theta^3(f \otimes 1_2)\, E_{18}^{14} \equiv \theta^{15}(G_{20} \otimes 1_2) \pmod{27}$$

in $S_{320}(\Gamma_0(36))$, using Theorem 6, it suffices to check coefficients of index $n \leq 1920$.

Sketch of Proof of Type II Congruences

Let $\ell > 3$ be an exceptional prime of Type II for an eta-quotient newform f. Then for all primes $p \nmid N\ell$, we have $a(p) \equiv 0 \pmod{\ell}$.

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To see that $\ell < 2k$, we argue by contradiction, and use $w_\ell(\theta^{\frac{\ell+1}{2}}(f \otimes 1_N)) = w_\ell(f \otimes 1_N))$ to obtain $\ell \in \{\pm 1\}$, which isn't possible.

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$$k + \ell + 1 = w_\ell(\theta f) = w_\ell(\theta^{\frac{\ell+1}{2}} f) = k + \frac{\ell+1}{2}(\ell-1) - \alpha(\ell-1)$$
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Hence, we conclude that

$$\ell \in \begin{cases} 2k-3 \text{ if } f \mid U_\ell \equiv 0 \pmod{\ell} \\ 2k-1 \text{ if } f \mid U_\ell \not\equiv 0 \pmod{\ell} \end{cases}$$

Thank you for your time.

Questions?