Signed partition numbers

Taylor Daniels

Purdue Univ.

March 14, 2024

MTU Seminar in Partition Theory, q-series, and Related Topics

Outline

1. An overview
2. Legendre-signed partitions
3. General circle method tools
4. The vanishing of $\mathfrak{p}(10j+2, \chi_5)$
5. (Bonus) Biasymptotics

Signed partitions

Definition

Let $f : \mathbb{N} \to \{0, \pm 1\}$. For any partition $\pi = (a_1, a_2, \ldots, a_k)$ of any n, set

$$f(\pi) := f(a_1)f(a_2)\cdots f(a_k),$$

so that $f(\pi) \in \{0, \pm 1\}$. The *f-signed partition numbers* are

$$\mathfrak{p}(n, f) := \sum_{\pi \in \Pi[n]} f(\pi),$$

the sum being taken over the set $\Pi[n]$ of all partitions of n.

First observations:

- $\mathfrak{p}(n, 1) = \mathfrak{p}(n)$, the "ordinary" partitions.
- $\mathfrak{p}(n, \mathbf{1}_A) = \mathfrak{p}_A(n)$ for $A \subset \mathbb{N}$. When clear, write $\mathfrak{p}(n, A)$.
- Certainly $-\mathfrak{p}(n, 1) \leq \mathfrak{p}(n, f) \leq \mathfrak{p}(n, 1)$.

Classical partition asymptotics

- (Hardy and Ramanujan, 1918)

$$\mathfrak{p}(n,1) \sim (4\sqrt{3})^{-1} \exp(\kappa\sqrt{n}), \qquad \kappa := \pi\sqrt{2/3}.$$

- (Erdős, 1942) Let $A \subset \mathbb{N}$; suppose $\gcd(A) = 1$ and $\delta_A = \lim_N \frac{|A \cap \{1,\ldots,N\}|}{N}$ exists. One has $\delta_A > 0$ if and only if

$$\log \mathfrak{p}(n,A) \sim \kappa\sqrt{\delta_A n}.$$

- $Q := \{n : n \text{ squarefree}\}$, then $\delta_Q = 6/\pi^2 \approx 0.601$, so

$$\log \mathfrak{p}(n,Q) \sim 2\sqrt{n}.$$

- (Roth and Szekeres, 1954) With $\mathbb{P}$ the set of primes, one has

$$\log \mathfrak{p}(n,\mathbb{P}) \sim \kappa\sqrt{2n/\log n}.$$

These sequences are all *asymptotic*.

Some "tricky" signed partition asymptotics

If $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ with distinct p_i, then

$$\lambda(n) = (-1)^{a_1 + \cdots + a_r} \qquad \text{and} \qquad \mu(n) = \begin{cases} (-1)^r & \text{if all } a_i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem (D., 2023)

Let $\kappa := \pi\sqrt{2/3}$. For all $\varepsilon > 0$, as $n \to \infty$ one has

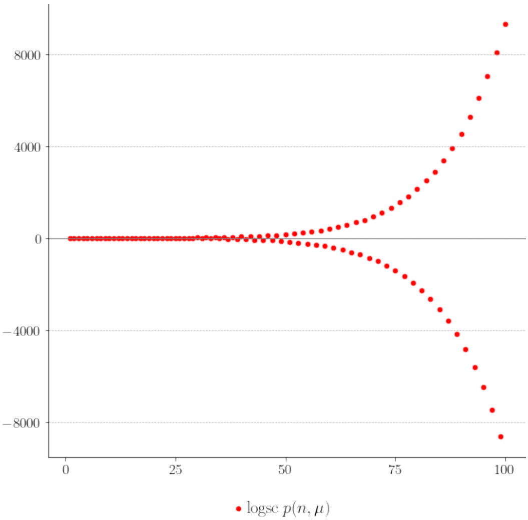
$$\mathfrak{p}(n, \mu) \ll e^{(1+\varepsilon)\sqrt{n}} \quad \textit{and} \quad \mathfrak{p}(n, \lambda) \ll e^{(\frac{1}{2}\kappa + \varepsilon)\sqrt{n}}.$$

In addition, for $n = 2k$, as $k \to \infty$ one has

$$\log \mathfrak{p}(2k, \mu) \sim \sqrt{2k} \qquad \textit{and} \qquad \log \mathfrak{p}(2k, \lambda) \sim \tfrac{1}{2}\kappa\sqrt{2k}.$$

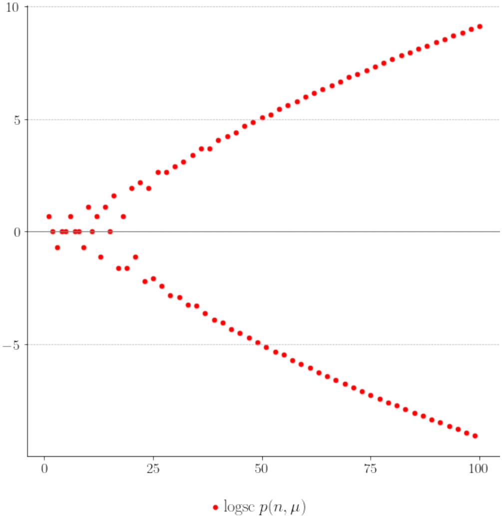
The behavior for both on odd n is delicate; more-so for λ.

Plot of $\mathfrak{p}(n, \mu)$ for $n \leq 100$



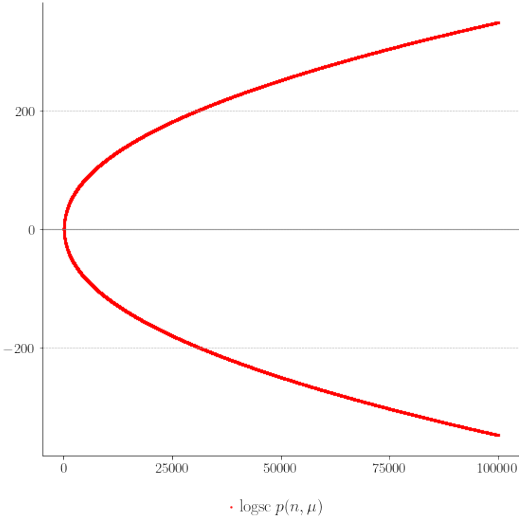
Plot of $\operatorname{logsc} \mathfrak{p}(n, \mu)$ for $n \leq 100$

$\operatorname{logsc}(x) := \operatorname{sgn}(x) \log(|x| + 1)$



Plot of $\operatorname{logsc} \mathfrak{p}(n, \mu)$ for $n \leq 10^5$

$\operatorname{logsc}(x) := \operatorname{sgn}(x) \log(|x| + 1)$



Outline

Legendre-signed partition numbers

Definition

Let $f : \mathbb{N} \to \{0, \pm 1\}$. For any partition $\pi = (a_1, a_2, \ldots, a_k)$ of any n, set

$$f(\pi) := f(a_1)f(a_2)\cdots f(a_k),$$

so that $f(\pi) \in \{0, \pm 1\}$. The *f-signed partition numbers* are

$$\mathfrak{p}(n, f) := \sum_{\pi \in \Pi[n]} f(\pi),$$

the sum being taken over the set $\Pi[n]$ of all partitions of n.

For an odd prime p, let $\chi_p(n)$ be the *Legendre symbol* $\left(\frac{n}{p}\right)$, namely

$$\chi_p(n) = \begin{cases} +1 & n \text{ is a quadratic residue } (\mathrm{mod}\ p), \\ -1 & n \text{ is a quadratic nonresidue } (\mathrm{mod}\ p), \\ 0 & p \mid n. \end{cases}$$

Ask: “Which $\mathfrak{p}(n, \chi_p)$ have ‘nice’ asymptotic behavior?”

Legendre-signed partition numbers

For an odd prime p, let $\chi_p(n)$ be the *Legendre symbol* $\left(\frac{n}{p}\right)$, namely

$$\chi_p(n) = \begin{cases} +1 & n \text{ is a quadratic residue} \pmod p, \\ -1 & n \text{ is a quadratic nonresidue} \pmod p, \\ 0 & p \mid n. \end{cases}$$

Ask: "Which $\mathfrak{p}(n, \chi_p)$ have 'nice' asymptotic behavior?"

$$\log \mathfrak{p}(n, 1) \sim \kappa\sqrt{n} \qquad \text{and} \qquad \log \mathfrak{p}(2k, \lambda) \sim \tfrac{1}{2}\kappa\sqrt{n}.$$

Theorem (D., 2024)

If p is an odd prime such that $p \neq 5$ and $p \not\equiv 1 \pmod 8$, then

$$\log \mathfrak{p}(n, \chi_p) \sim \tfrac{1}{2}\kappa\sqrt{\left(1 - \tfrac{1}{p}\right)n} \qquad \left(\kappa = \pi\sqrt{2/3}\right).$$

These sequences are *asymptotic*.

Finer asymptotics

Keep $\kappa = \pi\sqrt{2/3}$ and note that $\mathfrak{p}(n,1) \asymp n^{-1}\exp(\kappa\sqrt{n})$.

Corollary [to next slide] (D., 2024)

If $p \equiv 5 \pmod 8$ and $p \neq 5$, then as $n \to \infty$ one has

$$\mathfrak{p}(n,\chi_p) \asymp n^{-3/4}\exp\left(\tfrac{1}{2}\kappa\sqrt{(1-\tfrac{1}{p})n}\right).$$

While $\mathfrak{p}(2k,\chi_p)$ for $p \equiv 1 \pmod 8$ behaves similarly, the case of odd inputs requires case-dependent consideration.

Theorem (D., 2024)

Let $L_1(\chi_p) = L(1,\chi_p)$. If $p \equiv 3 \pmod 4$, then as $n \to \infty$ one has

$$\mathfrak{p}(n,\chi_p) \asymp n^{\sqrt{p}L_1(\chi_p)/4\pi - 3/4}\exp\left(\tfrac{1}{2}\kappa\sqrt{(1-\tfrac{1}{p})n}\right).$$

Why is $p = 5$ so special? A quick look at circle method

For $f : \mathbb{N} \to \mathbb{C}$ with $|f| \leq 1$ let

$$\Phi(z, f) := \prod_{n=1}^{\infty} (1 - f(n) z^n)^{-1} = \sum_{n=0}^{\infty} \mathfrak{p}(n, f) z^n \qquad (|z| < 1).$$

Compare with

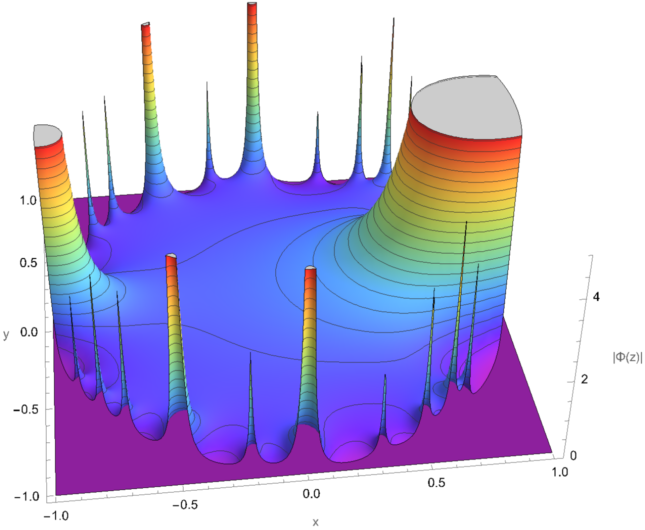
$$\prod_{n=1}^{\infty} (1 - z^n)^{-f(n)} \qquad \text{“weighted partition numbers”}.$$

By Cauchy's theorem we have

$$\mathfrak{p}(n, f) = \frac{1}{2\pi i} \int_{|z| = \rho} \Phi(z, f) z^{-n-1} \, dz$$

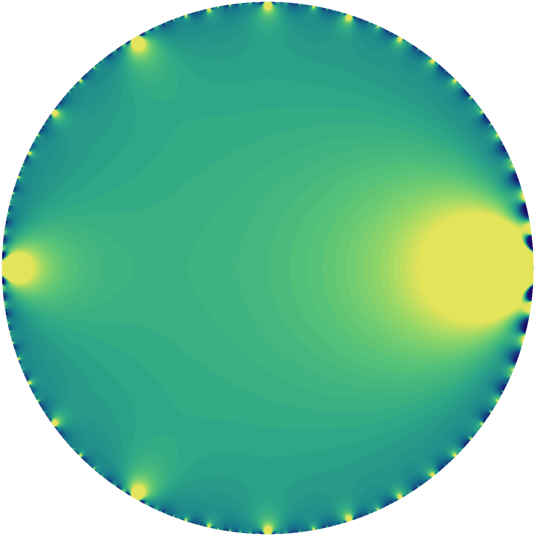
for all $0 < \rho < 1$.

A plot of $|\Phi(z, 1)|$ [a finite truncation of-]

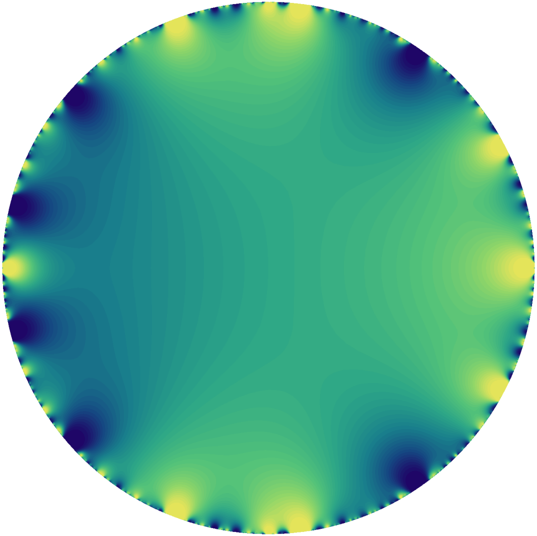


Based on a figure from Flajolet and Sedgewick's *Analytic Combinatorics*

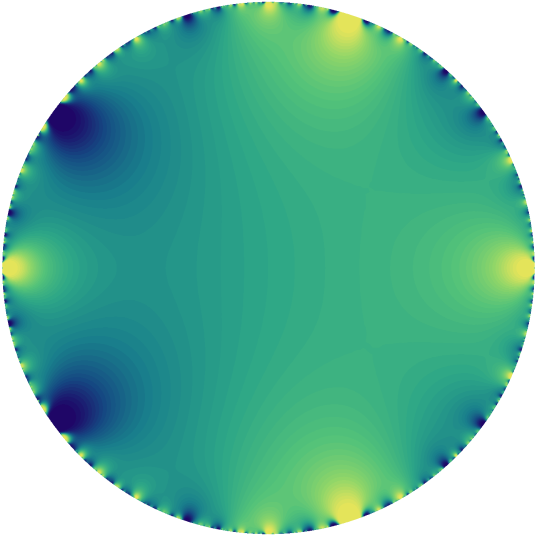
A plot of $|\Phi(z,1)|$



A plot of $|\Phi(z, \chi_{13})|$



A plot of $|\Phi(z, \chi_5)|$



A more precise formula when $p \equiv 1 \pmod 4$

Again let $L_1(\chi_p) = L(1, \chi_p)$. Recall

$$\mathfrak{p}(n, 1) = (4\sqrt{3})^{-1} n^{-1} \exp(\kappa\sqrt{n}) \Big[1 + O(n^{-1/5})\Big].$$

Theorem (D., 2024)

Let $p \neq 5$ and suppose that $p \equiv 1 \pmod 4$. As $n \to \infty$ one has

$$\mathfrak{p}(n, \chi_p) = \mathfrak{a}_p n^{-3/4} \exp\Big(\tfrac{1}{2}\kappa\sqrt{(1 - \tfrac{1}{p})n}\Big) \Big[1 + (-1)^n \mathfrak{b}_p + O(n^{-1/5})\Big],$$

where $\kappa = \pi\sqrt{2/3}$,

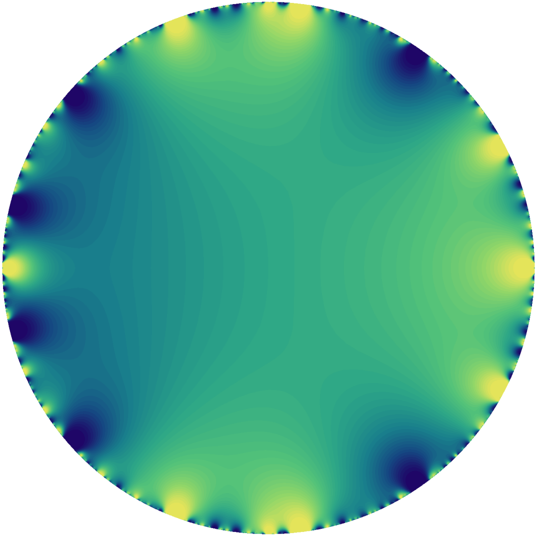
$$\mathfrak{a}_p = 2^{-7/4} 3^{-1/4} (p^{-1} - p^{-2})^{1/4} \exp(\tfrac{1}{4}\sqrt{p} L_1(\chi_p)),$$

and

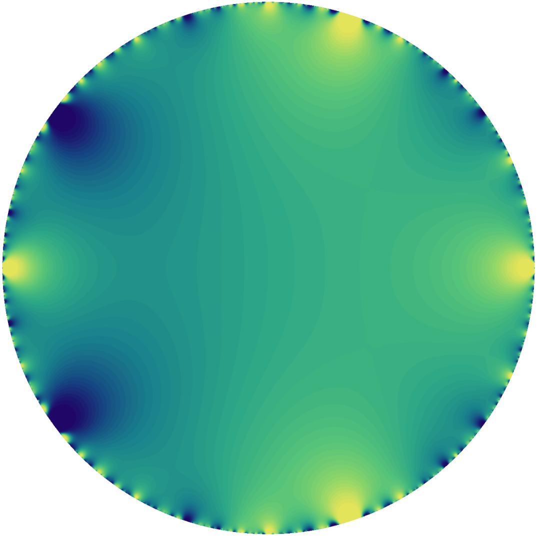
$$\mathfrak{b}_p = \begin{cases} 1 & p \equiv 1 \pmod 8, \\ \exp(-\sqrt{p} L_1(\chi_p)) & p \equiv 5 \pmod 8 \text{ and } p \neq 5. \end{cases}$$

$L_1(\chi_p) > 0$ for all p, so $\mathfrak{b}_p < 1$ when $p \equiv 5 \pmod 8$ and $p \neq 5$.

A plot of $|\Phi(z, \chi_{13})|$



A plot of $|\Phi(z, \chi_5)|$



Why is $p = 5$ so special?

- We "rank" singularities based on their exponential factor $\exp(c\kappa\sqrt{n})$.
- Most $\Phi(z, \chi_p)$ have two *principal* singularities. $\Phi(z, \chi_5)$ has four.

Theorem (D., 2024)

Let $\kappa = \pi\sqrt{2/3}$. As $n \to \infty$ one has

$$\mathfrak{p}(n, \chi_5) = \mathfrak{a}_5 n^{-\frac{3}{4}} \exp\left(\tfrac{1}{2}\kappa\sqrt{\tfrac{4}{5}n}\right) \times \left[1 + (-1)^n \mathfrak{b}_5 + \mathfrak{d}_5 \cos\left(\tfrac{2\pi}{5}n - \tfrac{\pi}{10}\right) + O\left(n^{-\frac{1}{5}}\right)\right],$$

$$\mathfrak{a}_5 = \left(\frac{3+\sqrt{5}}{960}\right)^{1/4}, \quad \mathfrak{b}_5 = \frac{3-\sqrt{5}}{2}, \quad \textit{and} \quad \mathfrak{d}_5 = \sqrt{2(5-\sqrt{5})}.$$

The "signed" term $\mathfrak{S}(n)$

$$\mathfrak{S}(n) := 1 + (-1)^n \left(\frac{3 - \sqrt{5}}{2} \right) + \sqrt{2(5 - \sqrt{5})} \cos\left(\tfrac{2\pi}{5} n - \tfrac{\pi}{10} \right),$$

$$\text{so that} \qquad \mathfrak{p}(n, \chi_5) = \mathfrak{a}_5 n^{-3/4} \exp\left(\tfrac{1}{2} \kappa \sqrt{\tfrac{4}{5} n} \right) \mathfrak{S}(n).$$

Computing the values of $\mathfrak{S}(n)$ for $1 \leq n \leq 10$, it is surprising to find that

$$\mathfrak{S}(2) = 0 \qquad \text{and} \qquad \mathfrak{S}(n) \neq 0 \quad \text{for } 1 \leq n \leq 10 \text{ with } n \neq 2.$$

Indeed, since

$$\cos\left(\frac{7\pi}{10} \right) = -\frac{1}{2} \sqrt{\frac{5 - \sqrt{5}}{2}},$$

we have

$$\mathfrak{S}(2) = \frac{5 - \sqrt{5}}{2} - \frac{\sqrt{(5 - \sqrt{5})^2}}{2} = 0.$$

Does this mean that $\mathfrak{p}(10m + 2, \chi_5) = 0$? There is an error term...

Outline

1. An overview
2. Legendre-signed partitions
3. General circle method tools
4. The vanishing of $\mathfrak{p}(10j + 2, \chi_5)$
5. (Bonus) Biasymptotics

Analytic tools; the circle method

A simple generating function is

$$\Phi(z,1) = \prod_{n=1}^{\infty}\left(\frac{1}{1-z^n}\right) = \sum_{n=0}^{\infty}\mathfrak{p}(n,1)z^n \qquad (|z|<1).$$

For $f : \mathbb{N} \to \mathbb{C}$ with $|f| \le 1$ let

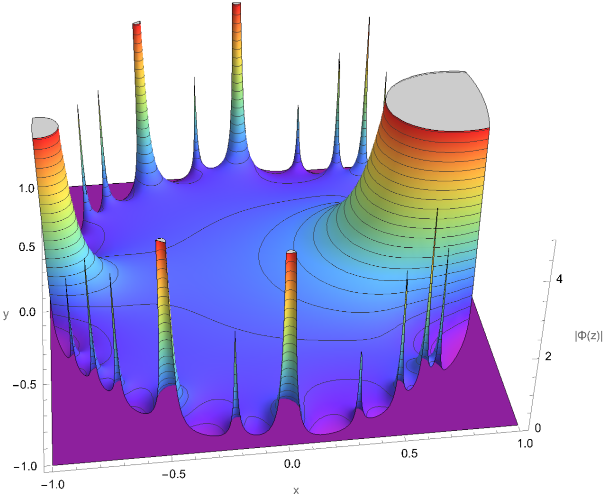
$$\Phi(z,f) := \prod_{n=1}^{\infty}\left(\frac{1}{1-f(n)z^n}\right) = \sum_{n=0}^{\infty}\mathfrak{p}(n,f)z^n \qquad (|z|<1).$$

By Cauchy's theorem we have

$$\mathfrak{p}(n,f) = \frac{1}{2\pi i}\int_{|z|=\rho}\Phi(z)z^{-n-1}\,dz$$

for all $0<\rho<1$.

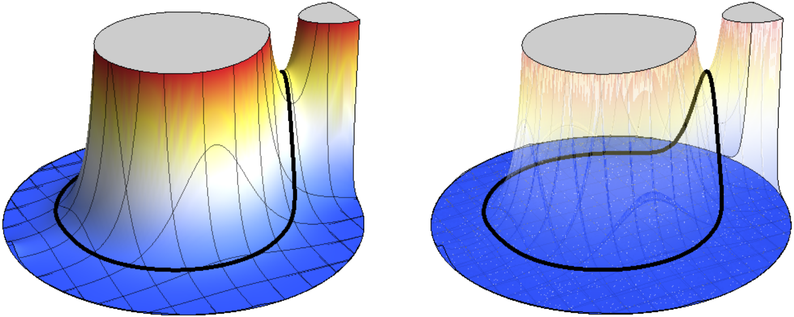
A plot of $|\Phi(z,1)|$



Based on a figure from Flajolet and Sedgewick's *Analytic Combinatorics*

The saddle point

Zooming in on $|\Phi(z)z^{-n-1}|$ with $n = 5$. We take the radius $\rho \approx 0.626$.



Dividing $[0, 1)$ into arcs

We now examine the integral(s)

$$\mathfrak{p}(n, f) = \frac{1}{2\pi i} \int_{|z|=\rho} \Phi(z) z^{-n-1}\, dz = \rho^{-n} \int_0^1 \Phi(\rho e(\alpha)) e(-n\alpha)\, d\alpha.$$

Want to split $[0, 1)$ into *principal*-, *major*-, and *minor*- "arcs".

- Principal arcs are the "dominant" arcs, usually near 0 and 1. For $\mathfrak{p}(n, \chi_p)$ the arc about $\alpha = 1/2$ is also principal.
- Major arcs are around reduced rationals $a/q \in [0, 1)$ with $q \leq Q$.
- Minor arcs are everything else (near a/q with $q > Q$). Error term.

$$\Psi(z, f) = \Psi(z) := \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} f^k(n) z^{nk}/k \qquad (|z| < 1)$$

$$\Phi(z) = \exp \Psi(z)$$

What do we need to study $\mathfrak{p}(n, f)$?

A "program" described by Gafni for studying $\mathfrak{p}(n, \mathbf{1}_A)$ which works for us:

- For the principal arcs, need to know about $\sum_n f(n) n^{-s}$.
- For the major arcs, need to know about "residue" sums

$$\sum_{n \equiv a \pmod{q}} f(n) \qquad (\text{with } (a, q) = 1).$$

- For the minor arcs, need to know about exponential ("Weyl") sums

$$\sum_{n \leq X} f(n) e(n\alpha).$$

Gafni's "program" is essentially $f(n) := \mathbf{1}_A$, giving

$$\sum_{n \in A} n^{-s}, \qquad \sum_{\substack{n \in A \\ n \equiv a \pmod{q}}} 1, \quad \text{and} \quad \sum_{n \in [1..X] \cap A} e(n\alpha).$$

A general minor arc result

Theorem (Montgomery and Vaughan, 1977)

Suppose that $|\alpha - a/q| \leq q^{-2}$, $(a, q) = 1$, *and* $2 \leq R \leq q \leq X/R$. *Then*

$$\sum_{n \leq X} f(n)e(n\alpha) \ll \frac{X}{\log X} + \frac{X \log^{3/2} R}{R^{1/2}}$$

uniformly for all multiplicative f *with* $|f| \leq 1$.

Theorem (D., 2024)

Suppose α *is such that: if* $|\alpha - a/q| \leq 1/(qX^{2/3})$ *and* $(a, q) = 1$, *then* $q > X^{1/3}$. *Writing* $\rho = e^{-1/X}$, *as* $X \to \infty$ *one has*

$$\Psi(\rho e(\alpha), f) \ll X/\log X$$

uniformly for multiplicative f *with* $|f| \leq 1$.

Outline

Two curious vanishings

Keep $\chi_5(n) = (\frac{n}{5})$, and for $\pi = (a_1, a_2, \ldots, a_k)$ let

$$\chi_5^\dagger(\pi) := (-1)^k \chi_5(a_1)\chi_5(a_2)\cdots\chi_5(a_k) = [-\chi_5](\pi).$$

Theorem (D., 2024)

One has

$$\begin{aligned} \mathfrak{p}(n,\chi_5) &= 0 \qquad n \equiv 2 \pmod{10}, \\ \mathfrak{p}(n,\chi_5^\dagger) &= 0 \qquad n \equiv 6 \pmod{10}. \end{aligned}$$

and

$$\begin{aligned} \mathfrak{p}(n,\chi_5) &= \mathfrak{p}(n,\chi_5^\dagger) \qquad n \equiv 0 \pmod{10}, \\ \mathfrak{p}(n,\chi_5) &= -\mathfrak{p}(n,\chi_5^\dagger) \qquad n \equiv 8 \pmod{10} \end{aligned}$$

q-series methods

To examine $\mathfrak{p}(n, \chi_5)$ and $\mathfrak{p}(n, \chi_5^\dagger)$, let

$$(z; q) = (z; q)_\infty = \prod_{n=0}^{\infty} (1 - zq^n),$$
$$(z_1, \ldots, z_m; q) = (z_1; q) \cdots (z_m; q),$$

and

$$X(q) := (q, -q^2, -q^3, q^4; q^5) \quad \text{and} \quad Y(q) := (-q, q^2, q^3, -q^4; q^5)$$

so that

$$\frac{1}{X(q)} = \sum_{n=0}^{\infty} \mathfrak{p}(n, \chi_5) q^n \quad \text{and} \quad \frac{1}{Y(q)} = \sum_{n=0}^{\infty} \mathfrak{p}(n, \chi_5^\dagger) q^n.$$

q-series setup

Again

$$X(q) := (q, -q^2, -q^3, q^4; q^5) \quad \text{and} \quad Y(q) := (-q, q^2, q^3, -q^4; q^5)$$

Euler's product

$$\varphi(q) := (q; q) \qquad \text{and} \qquad \varphi_m := \varphi(q^m).$$

Using these, we have

$$X(q)Y(q) = \varphi(q^{10})/\varphi(q^2) = \varphi_{10}/\varphi_2,$$

and thus

$$\frac{\varphi_{10}^3}{X(q)} = \frac{Y(q)\varphi_{10}^4}{\varphi_2} \qquad \text{and} \qquad \frac{\varphi_{10}^3}{Y(q)} = \frac{X(q)\varphi_{10}^4}{\varphi_2}.$$

From here, we need 10-dissections of $Y(q)\varphi_{10}^4$, $X(q)\varphi_{10}^4$, and $1/\varphi_2$.

An *m-dissection* of $f(q)$ is

$$f(q) = q^0 f_0(q^m) + q^1 f_1(q^m) + \cdots + q^{m-1} f_{m-1}(q^m).$$

The Rogers-Ramanujan continued fraction

$$R(q) := \frac{(q^1, q^4; q^5)}{(q^2, q^3; q^5)}.$$

Let $R_5 := R(q^5)$. From (Hirschhorn, eq. 8.4.4) we have

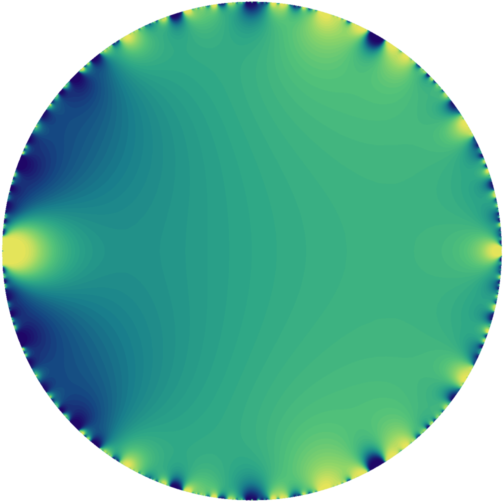
$$\frac{1}{\varphi} = \frac{\varphi_{25}^5}{\varphi_5^6}\Big(R_5^{-4} + qR_5^{-3} + 2q^2R_5^{-2} + 3q^3R_5^{-1} + 5q^4 - 3q^5R_5 + 2q^6R_5^2 - q^7R_5^3 + q^8R_5^4\Big),$$

then substitute q with q^2.

Outline

Back to $\mathfrak{p}(n, \mu)$ and $\mathfrak{p}(n, \lambda)$

A plot of $|\Phi(z, \mu)|$.



Back to μ and λ

Recall

$$\mathfrak{p}(n,f) = \sum_{\pi \in \Pi[n]} f(\pi) \qquad \text{with} \qquad f(\pi) = f(a_1)f(a_2)\cdots f(a_k).$$

If $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ with distinct p_i, then

$$\lambda(n) = (-1)^{a_1+\cdots+a_r} \qquad \text{and} \qquad \mu(n) = \begin{cases} (-1)^r & \text{if all } a_i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem (D., 2023)

Let $\kappa := \pi\sqrt{2/3}$. For all $\varepsilon > 0$, as $n \to \infty$ one has

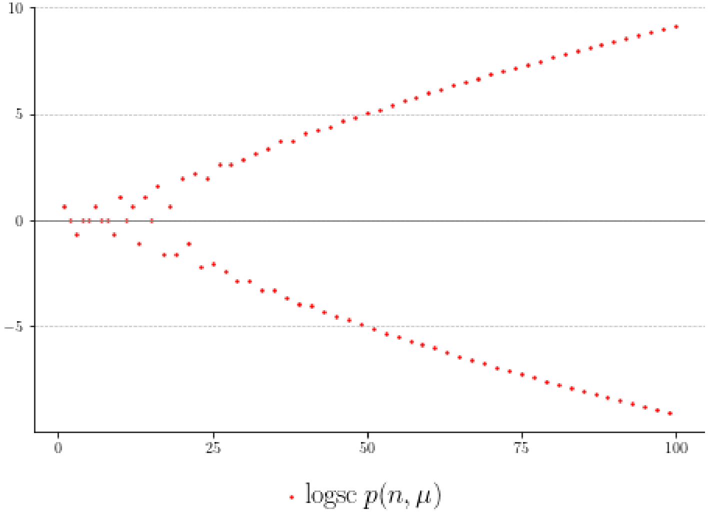
$$\mathfrak{p}(n,\mu) \ll e^{(1+\varepsilon)\sqrt{n}} \quad \textit{and} \quad \mathfrak{p}(n,\lambda) \ll e^{(\frac{1}{2}\kappa+\varepsilon)\sqrt{n}}.$$

In addition, for $n = 2k$, as $k \to \infty$ one has

$$\log \mathfrak{p}(2k,\mu) \sim \sqrt{2k} \qquad \textit{and} \qquad \log \mathfrak{p}(2k,\lambda) \sim \tfrac{1}{2}\kappa\sqrt{2k}.$$

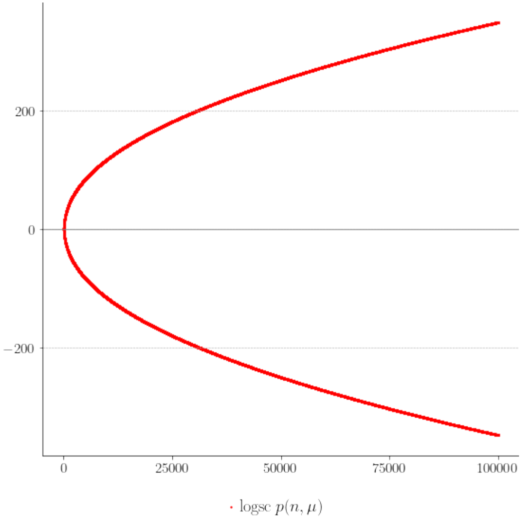
Plots for $n \leq 250$

$\operatorname{logsc} x := \operatorname{sgn}(x) \log(|x| + 1)$



Plots for $n \leq 10^5$

$\operatorname{logsc} x := \operatorname{sgn}(x) \log(|x| + 1)$



“Biasymptotics”

The question of odd n is delicate, and is explored in its own paper.

Assumption

Suppose $\sup\{\mathrm{Re}(\rho) : \zeta(\rho) = 0\} < 1$. This is almost certainly unnecessary.

Theorem (Rough statement)

Under above assumption $\mathfrak{p}(n, \mu)$ infinitely oscillates between arbitrarily long streaks where $\mathfrak{p}(n, \mu) \approx e^{\sqrt{n}+o(\sqrt{n})}$ and arbitrarily long streaks of behaving $(-1)^n e^{\sqrt{n}+o(\sqrt{n})}$. We say $\mathfrak{p}(n, \mu)$ is biasymptotic.

Some difficulties:

- Some $\mathfrak{p}(n, \mu) = 0$.
- For small n, say $50 \le n \le 10000$, one has $\mathfrak{p}(n, \mu) \approx (-1)^n e^{\sqrt{n}}$.

What about $\mathfrak{p}(n, \lambda)$?

If either Riemann or Simplicity fail, then $\mathfrak{p}(n, \lambda)$ is biasymptotic too, but with exponential term $\exp(\frac{1}{2}\kappa\sqrt{n})$.

Theorem (D., 2023)

Suppose Riemann and Simplicity hold, and additionally that $|\zeta'(\rho)| \leq C|\rho|$ for all zeros $\rho = \frac{1}{2} + i\gamma$. There is some $\mathfrak{c} > 0$ such that: if $C < \mathfrak{c}$, then

$$(-1)^n \operatorname{logsc} \mathfrak{p}(n, \lambda) \sim \tfrac{1}{2}\kappa\sqrt{n} \qquad (\kappa = \pi\sqrt{2/3}).$$

Namely $\mathfrak{p}(n, \lambda)$ is "asymptotic" and $\mathfrak{p}(n, \lambda) = (-1)^n \exp(\frac{1}{2}\kappa\sqrt{n} + o(\sqrt{n}))$.

Moreover, it holds that $\mathfrak{c} > 10^{17881}$.

The quantity 10^{17881} is nowhere near "optimal".

How to "efficiently" compute $\mathfrak{p}(n, f)$

Fact

With $\mathfrak{p}(0) := 1$, for all n one has

$$\mathfrak{p}(n) = \frac{1}{n}\sum_{k=0}^{n-1} \mathfrak{p}(k)\sigma(n-k), \qquad \text{where} \qquad \sigma(k) = \sum_{d|k} d.$$

Easy generalization

$$\mathfrak{p}(n, f) = \frac{1}{n}\sum_{k=0}^{n-1} \mathfrak{p}(k, f)\mathfrak{S}(n-k, f), \quad \text{where} \quad \mathfrak{S}(k, f) := \sum_{d|k} f(d)^{k/d} d.$$

Recursively computing $\mathfrak{p}(n, f)$ using GP/Pari takes about 9-12 seconds (on my computer) for $n \leq 10^4$; about 11 minutes for $n \leq 10^5$.

My thanks for attending!