Vanishings of Legendre-signed partitions and related q-series on arithmetic progressions

Taylor Daniels

Purdue University

March 6, 2025

MTU seminar on partitions and q-series

Basic Definitions

Conventions

Throughout, let $p > 2$ be prime and let

$$\chi = \chi_p = \left(\frac{\cdot}{p}\right)$$

be the Legendre symbol; namely, for a coprime to p let

$$\chi(a) = \begin{cases} 1 & x^2 \equiv a \pmod{p} \text{ is soluble,} \\ -1 & x^2 \equiv a \pmod{p} \text{ is nonsoluble,} \end{cases}$$

and set $\chi(a) = 0$ when $p \mid a$.

Basic Definitions

Conventions

Throughout, let $p > 2$ be prime and let

$$\chi = \chi_p = \left(\frac{\cdot}{p}\right)$$

be the Legendre symbol; namely, for a coprime to p let

$$\chi(a) = \begin{cases} 1 & x^2 \equiv a \pmod{p} \text{ is soluble,} \\ -1 & x^2 \equiv a \pmod{p} \text{ is nonsoluble,} \end{cases}$$

and set $\chi(a) = 0$ when $p \mid a$.

Definition

Partitions of n are $\pi = (a_1, \ldots, a_k)$ with $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$ and $a_1 + \cdots + a_k = n$. No restriction on k.

Legendre-signed partitions

Definition (Partition-focused)

For any partition $\pi = (a_1, \ldots, a_k)$ of any n, set

$$\chi(\pi) := \chi(a_1)\chi(a_2)\cdots\chi(a_k),$$

Legendre-signed partitions

Definition (Partition-focused)

For any partition $\pi = (a_1, \ldots, a_k)$ of any n, set

$$\chi(\pi) := \chi(a_1)\chi(a_2)\cdots\chi(a_k),$$
$$\chi_p^\dagger(\pi) := [-\chi_p](\pi) = (-1)^k\chi_p(\pi),$$

so $\chi_p(\pi), \chi_p^\dagger(\pi) \in \{0, \pm 1\}$.

Legendre-signed partitions

Definition (Partition-focused)

For any partition $\pi = (a_1, \ldots, a_k)$ of any n, set

$$\chi(\pi) := \chi(a_1)\chi(a_2)\cdots\chi(a_k),$$
$$\chi_p^\dagger(\pi) := [-\chi_p](\pi) = (-1)^k\chi_p(\pi),$$

so $\chi_p(\pi), \chi_p^\dagger(\pi) \in \{0, \pm 1\}$. The *Legendre-signed partition numbers* are

$$\mathfrak{p}(n, \chi_p) := \sum_{\pi \in \Pi[n]} \chi_p(\pi), \qquad \text{and} \qquad \mathfrak{p}(n, \chi_p^\dagger) := \sum_{\pi \in \Pi[n]} \chi_p^\dagger(\pi),$$

with $\Pi[n]$ the set of all partitions of n.

Legendre-signed partitions

Definition (Partition-focused)

For any partition $\pi = (a_1, \ldots, a_k)$ of any n, set

$$\chi(\pi) := \chi(a_1)\chi(a_2)\cdots\chi(a_k),$$
$$\chi_p^\dagger(\pi) := [-\chi_p](\pi) = (-1)^k \chi_p(\pi),$$

so $\chi_p(\pi), \chi_p^\dagger(\pi) \in \{0, \pm 1\}$. The *Legendre-signed partition numbers* are

$$\mathfrak{p}(n, \chi_p) := \sum_{\pi \in \Pi[n]} \chi_p(\pi), \qquad \text{and} \qquad \mathfrak{p}(n, \chi_p^\dagger) := \sum_{\pi \in \Pi[n]} \chi_p^\dagger(\pi),$$

with $\Pi[n]$ the set of all partitions of n.

- $\mathfrak{p}(n) = \mathfrak{p}(n, 1)$.

Legendre-signed partitions

Definition (Partition-focused)

For any partition $\pi = (a_1, \ldots, a_k)$ of any n, set

$$\chi(\pi) := \chi(a_1)\chi(a_2)\cdots\chi(a_k),$$
$$\chi_p^\dagger(\pi) := [-\chi_p](\pi) = (-1)^k \chi_p(\pi),$$

so $\chi_p(\pi), \chi_p^\dagger(\pi) \in \{0, \pm 1\}$. The *Legendre-signed partition numbers* are

$$\mathfrak{p}(n, \chi_p) := \sum_{\pi \in \Pi[n]} \chi_p(\pi), \qquad \text{and} \qquad \mathfrak{p}(n, \chi_p^\dagger) := \sum_{\pi \in \Pi[n]} \chi_p^\dagger(\pi),$$

with $\Pi[n]$ the set of all partitions of n.

- $\mathfrak{p}(n) = \mathfrak{p}(n, 1)$.
- $\mathfrak{p}(n, 1_A)$ are A-restricted partitions (e.g., $A = \mathbb{P}$).

Legendre-signed partitions

Definition (Partition-focused)

For any partition $\pi = (a_1, \ldots, a_k)$ of any n, set

$$\chi(\pi) := \chi(a_1)\chi(a_2)\cdots\chi(a_k),$$
$$\chi_p^\dagger(\pi) := [-\chi_p](\pi) = (-1)^k\chi_p(\pi),$$

so $\chi_p(\pi), \chi_p^\dagger(\pi) \in \{0, \pm 1\}$. The *Legendre-signed partition numbers* are

$$\mathfrak{p}(n, \chi_p) := \sum_{\pi \in \Pi[n]} \chi_p(\pi), \qquad \text{and} \qquad \mathfrak{p}(n, \chi_p^\dagger) := \sum_{\pi \in \Pi[n]} \chi_p^\dagger(\pi),$$

with $\Pi[n]$ the set of all partitions of n.

- $\mathfrak{p}(n) = \mathfrak{p}(n, 1)$.
- $\mathfrak{p}(n, 1_A)$ are A-restricted partitions (e.g., $A = \mathbb{P}$).
- Clearly $-\mathfrak{p}(n) \leq \mathfrak{p}(n, \chi_p) \leq \mathfrak{p}(n)$.

Legendre-signed partitions (cont.)

q-Series Notation

$$(z;q)_\infty = \prod_{m=0}^{\infty}(1 - zq^m) \quad \text{and} \quad (z_1,\ldots,z_r;q)_\infty = \prod_{a=1}^{r}(z_a;q)_\infty.$$

Legendre-signed partitions (cont.)

q-Series Notation

$$(z;q)_\infty = \prod_{m=0}^{\infty}(1 - zq^m) \quad \text{and} \quad (z_1, \ldots, z_r; q)_\infty = \prod_{a=1}^{r}(z_a; q)_\infty .$$

Definition (q-series-focused)

$$\prod_{a=1}^{p-1}(+\chi(a)q^a; q^p)_\infty^{-1} = \sum_{n=0}^{\infty}\mathfrak{p}(n, \chi_p)q^n$$

$$\prod_{a=1}^{p-1}(-\chi(a)q^a; q^p)_\infty^{-1} = \sum_{n=0}^{\infty}\mathfrak{p}(n, \chi_p^\dagger)q^n$$

Legendre-signed partitions (cont.)

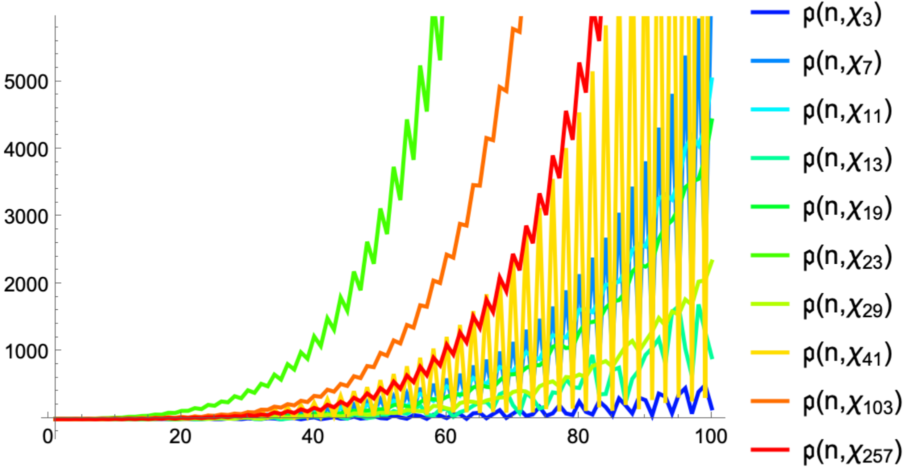
Definition (q-series-focused)

$$\prod_{a=1}^{p-1}(+\chi(a)q^a;q^p)_\infty^{-1} = \sum_{n=0}^{\infty}\mathfrak{p}(n,\chi_p)q^n$$
$$\prod_{a=1}^{p-1}(-\chi(a)q^a;q^p)_\infty^{-1} = \sum_{n=0}^{\infty}\mathfrak{p}(n,\chi_p^\dagger)q^n$$

Examples for $p = 5$

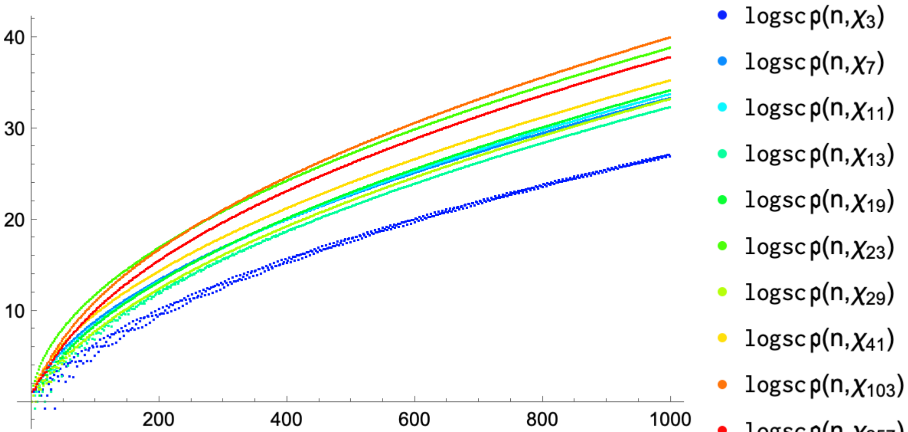
$$(+q^1,-q^2,-q^3,+q^4;q^5)_\infty^{-1} = \sum_{n=0}^{\infty}\mathfrak{p}(n,\chi_5)q^n \qquad (\chi_5\text{-signed}),$$
$$(-q^1,+q^2,+q^3,-q^4;q^5)_\infty^{-1} = \sum_{n=0}^{\infty}\mathfrak{p}(n,\chi_5^\dagger)q^n \qquad ((-\chi_5)\text{-signed}).$$

Some plots for small n and p



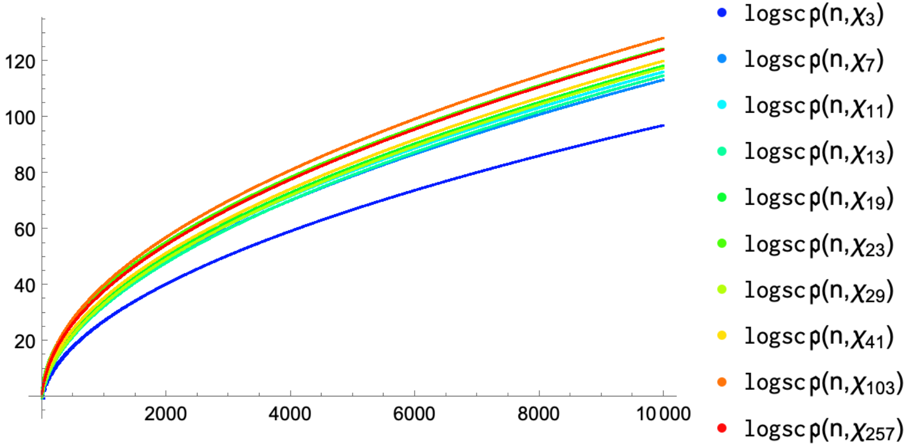
Some plots for small n and p

$\operatorname{logsc} x := \operatorname{sgn}(x) \log(|x| + 1)$



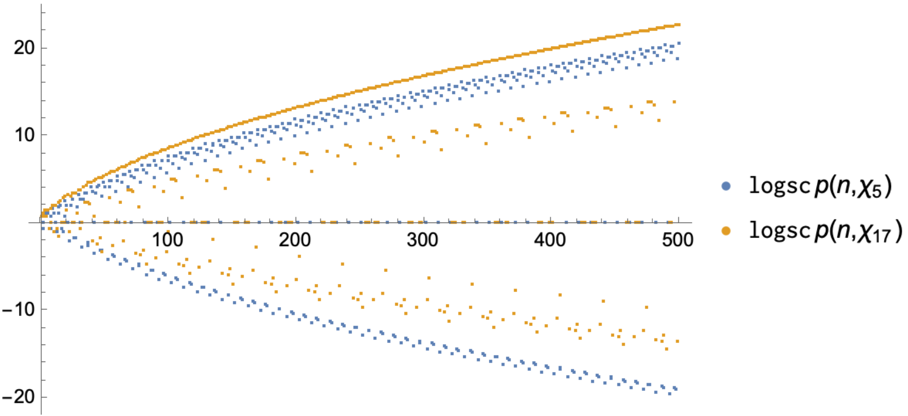
Some plots for small n and p

$$\operatorname{logsc} x := \operatorname{sgn}(x) \log(|x| + 1)$$



Two odd cases

$\operatorname{logsc} x := \operatorname{sgn}(x) \log(|x| + 1)$



Some observations

- The sequences $(\mathfrak{p}(n,\chi_5))_{\mathbb{N}}$ and $(\mathfrak{p}(n,\chi_{17}))_{\mathbb{N}}$ seem to separate into subsequences (arithmetic progressions $(\mathrm{mod}\ 10)$ and $(\mathrm{mod}\ 34)$, respectively) with repeating "value-trends".

Some observations

- The sequences $(\mathfrak{p}(n, \chi_5))_{\mathbb{N}}$ and $(\mathfrak{p}(n, \chi_{17}))_{\mathbb{N}}$ seem to separate into subsequences (arithmetic progressions $(\mathrm{mod}\, 10)$ and $(\mathrm{mod}\, 34)$, respectively) with repeating "value-trends".
- In particular, there are some arithmetic progressions on which they are identically 0.

Some observations

- The sequences $(\mathfrak{p}(n, \chi_5))_{\mathbb{N}}$ and $(\mathfrak{p}(n, \chi_{17}))_{\mathbb{N}}$ seem to separate into subsequences (arithmetic progressions $(\mathrm{mod}\, 10)$ and $(\mathrm{mod}\, 34)$, respectively) with repeating "value-trends".
- In particular, there are some arithmetic progressions on which they are identically 0.
- The other primes' graphs don't seem to have this trend.

Some observations

- The sequences $(\mathfrak{p}(n, \chi_5))_{\mathbb{N}}$ and $(\mathfrak{p}(n, \chi_{17}))_{\mathbb{N}}$ seem to separate into subsequences (arithmetic progressions $(\bmod 10)$ and $(\bmod 34)$, respectively) with repeating "value-trends".
- In particular, there are some arithmetic progressions on which they are identically 0.
- The other primes' graphs don't seem to have this trend.

Questions

1. Do these "vanishings on arithmetic progressions" continue indefinitely for $\mathfrak{p}(n, \chi_5)$ and $\mathfrak{p}(n, \chi_{17})$?

Some observations

- The sequences $(\mathfrak{p}(n, \chi_5))_{\mathbb{N}}$ and $(\mathfrak{p}(n, \chi_{17}))_{\mathbb{N}}$ seem to separate into subsequences (arithmetic progressions $(\mathrm{mod}\ 10)$ and $(\mathrm{mod}\ 34)$, respectively) with repeating "value-trends".
- In particular, there are some arithmetic progressions on which they are identically 0.
- The other primes' graphs don't seem to have this trend.

Questions

1. Do these "vanishings on arithmetic progressions" continue indefinitely for $\mathfrak{p}(n, \chi_5)$ and $\mathfrak{p}(n, \chi_{17})$?
2. For which other primes does this kind of vanishing occur?

What happens "most" of the time?

Theorem (D, 2024)

If $p \neq 2, 5$ *and* $p \not\equiv 1 \pmod 8$, *then* $\mathfrak{p}(n, \chi_p) \to \infty$ *as* $n \to \infty$. *More precisely, for these* p,

$$\mathfrak{p}(n, \chi_p) \asymp n^{c_p} \exp\left(\tfrac{1}{2}\kappa\sqrt{(1 - \tfrac{1}{p})n}\right) \qquad (\kappa = \pi\sqrt{2/3}).$$

What happens "most" of the time?

Theorem (D, 2024)

If $p \neq 2, 5$ and $p \not\equiv 1 \pmod 8$, then $\mathfrak{p}(n, \chi_p) \to \infty$ as $n \to \infty$. More precisely, for these p,

$$\mathfrak{p}(n, \chi_p) \asymp n^{c_p} \exp\left(\tfrac{1}{2}\kappa\sqrt{(1 - \tfrac{1}{p})n}\right) \qquad (\kappa = \pi\sqrt{2/3}).$$

In addition, when $p \equiv 1 \pmod 8$, one has $\mathfrak{p}(2m, \chi_p) \to \infty$ as $m \to \infty$.

What happens "most" of the time?

Theorem (D, 2024)

If $p \neq 2, 5$ and $p \not\equiv 1 \pmod 8$, then $\mathfrak{p}(n, \chi_p) \to \infty$ as $n \to \infty$. More precisely, for these p,

$$\mathfrak{p}(n, \chi_p) \asymp n^{c_p} \exp\left(\tfrac{1}{2}\kappa\sqrt{(1 - \tfrac{1}{p})n}\right) \qquad (\kappa = \pi\sqrt{2/3}).$$

In addition, when $p \equiv 1 \pmod 8$, one has $\mathfrak{p}(2m, \chi_p) \to \infty$ as $m \to \infty$.

1) (Hardy and Ramanujan, 1918)

$$\log \mathfrak{p}(n, 1) \sim \kappa\sqrt{n}$$

What happens "most" of the time?

Theorem (D, 2024)

If $p \neq 2, 5$ and $p \not\equiv 1 \pmod 8$, then $\mathfrak{p}(n, \chi_p) \to \infty$ as $n \to \infty$. More precisely, for these p,

$$\mathfrak{p}(n, \chi_p) \asymp n^{c_p} \exp\left(\tfrac{1}{2}\kappa\sqrt{(1 - \tfrac{1}{p})n}\right) \qquad (\kappa = \pi\sqrt{2/3}).$$

In addition, when $p \equiv 1 \pmod 8$, one has $\mathfrak{p}(2m, \chi_p) \to \infty$ as $m \to \infty$.

1) (Hardy and Ramanujan, 1918)

$$\log \mathfrak{p}(n, 1) \sim \kappa\sqrt{n}$$

2) (Erdős, 1942)

$$\log \mathfrak{p}(n, 1_A) \sim \kappa\sqrt{\delta_A n} \qquad (\delta_A = \lim_N |A \cap [N]|/N > 0)$$

What happens "most" of the time?

Theorem (D, 2024)

If $p \neq 2, 5$ and $p \not\equiv 1 \pmod 8$, then $\mathfrak{p}(n, \chi_p) \to \infty$ as $n \to \infty$. More precisely, for these p,

$$\mathfrak{p}(n, \chi_p) \asymp n^{c_p} \exp\left(\tfrac{1}{2}\kappa\sqrt{(1 - \tfrac{1}{p})n}\right) \qquad (\kappa = \pi\sqrt{2/3}).$$

In addition, when $p \equiv 1 \pmod 8$, one has $\mathfrak{p}(2m, \chi_p) \to \infty$ as $m \to \infty$.

1) (Hardy and Ramanujan, 1918)

$$\log \mathfrak{p}(n, 1) \sim \kappa\sqrt{n}$$

2) (Erdős, 1942)

$$\log \mathfrak{p}(n, 1_A) \sim \kappa\sqrt{\delta_A n} \qquad (\delta_A = \lim_N |A \cap [N]|/N > 0)$$

3) (Roth and Szekeres, 1954)

$$\log \mathfrak{p}(n, 1_{\mathbb{P}}) \sim \kappa\sqrt{2n/\log n}.$$

What happens "most" of the time?

Theorem (D, 2024)

If $p \neq 2, 5$ *and* $p \not\equiv 1 \pmod 8$, *then* $\mathfrak{p}(n, \chi_p) \to \infty$ *as* $n \to \infty$. *More precisely, for these* p,

$$\mathfrak{p}(n, \chi_p) \asymp n^{c_p} \exp\left(\tfrac{1}{2}\kappa\sqrt{(1 - \tfrac{1}{p})n}\right) \qquad (\kappa = \pi\sqrt{2/3}).$$

In addition, when $p \equiv 1 \pmod 8$, *one has* $\mathfrak{p}(2m, \chi_p) \to \infty$ *as* $m \to \infty$.

Proof Method

Use the Hardy-Littlewood circle method to compute the first- and second-order asymptotic terms, à-la Vaughan's (2008) work on partitions into prime numbers (i.e., $\mathfrak{p}(n, 1_{\mathbb{P}})$).

Vanishings for $p = 5$

Recall that for $\pi = (a_1, \ldots, a_k)$, we set

$$\chi_5(\pi) = \chi_5(a_1) \cdots \chi_5(a_k),$$
$$\chi_5^\dagger(\pi) = (-1)^k \chi_5(a_1) \cdots \chi_5(a_k).$$

Vanishings for $p = 5$

Recall that for $\pi = (a_1, \ldots, a_k)$, we set

$$\chi_5(\pi) = \chi_5(a_1) \cdots \chi_5(a_k),$$
$$\chi_5^\dagger(\pi) = (-1)^k \chi_5(a_1) \cdots \chi_5(a_k).$$

Theorem (D, 2024)

One has

$$\mathfrak{p}(n, \chi_5) = 0 \qquad \textit{for all } n \equiv 2 \pmod{10}$$
$$\mathfrak{p}(n, \chi_5^\dagger) = 0 \qquad \textit{for all } n \equiv 6 \pmod{10}.$$

Vanishings of $\mathfrak{p}(n, \chi_{17})$

Theorem (D, 2025+)

One has

$$\mathfrak{p}(n, \chi_{17}) = 0 \qquad \text{for all } n \equiv 17, 19, 25, 27 \pmod{34},$$

$$\mathfrak{p}(n, \chi_{17}^{\dagger}) = 0 \qquad \text{for all } n \equiv 11, 15, 29, 33 \pmod{34}.$$

Vanishings of $\mathfrak{p}(n, \chi_{17})$

Theorem (D, 2025+)

One has

$$\mathfrak{p}(n, \chi_{17}) = 0 \qquad \textit{for all } n \equiv 17, 19, 25, 27 \pmod{34},$$
$$\mathfrak{p}(n, \chi_{17}^{\dagger}) = 0 \qquad \textit{for all } n \equiv 11, 15, 29, 33 \pmod{34}.$$

The patterns

One has $\mathfrak{p}(n, \chi_{17}) = 0$ when

- n is odd;
- $1 - 24n$ is congruent to a quartic residue $\pmod{17}$.

Vanishings of $\mathfrak{p}(n, \chi_{17})$

Theorem (D, 2025+)

One has

$$\mathfrak{p}(n, \chi_{17}) = 0 \qquad \text{for all } n \equiv 17, 19, 25, 27 \pmod{34},$$
$$\mathfrak{p}(n, \chi_{17}^{\dagger}) = 0 \qquad \text{for all } n \equiv 11, 15, 29, 33 \pmod{34}.$$

The patterns

One has $\mathfrak{p}(n, \chi_{17}) = 0$ when

- n is odd;
- $1 - 24n$ is congruent to a quartic residue $\pmod{17}$.

One has $\mathfrak{p}(n, \chi_{17}^{\dagger}) = 0$ when

- n is odd;
- $1 - 24n$ is congruent to a quadratic-nonquartic residue $\pmod{17}$.

The big idea (for $p = 17$)

$$\mathfrak{p}(n,\chi_p) = \sum_{\substack{k=1 \\ 2\nmid k,\, p\nmid k}}^{\infty} \frac{1}{k}\Big\{ \qquad\qquad \Big\} F(n,k) + \sum_{\substack{k=1 \\ 2\mid k,\, p\nmid k}}^{\infty} \frac{1}{k}\Big\{ \qquad\qquad \Big\} F(n,k)$$

$$+ \sum_{\substack{k=1 \\ 4\mid k,\, p\nmid k}}^{\infty} \frac{1}{k}\Big\{ \qquad\qquad \Big\} F(n,k)$$

$$+ \sum_{\substack{k=1 \\ 2\nmid k,\, p\mid k}}^{\infty} \sum_{m=0}^{2} \frac{1}{k}\Big\{ \qquad\qquad\qquad \Big\} G(n,m,k).$$

Remark

The terms with $2p \mid k$ do not contribute asymptotic terms.

The big idea (for $p = 17$)

$$\mathfrak{p}(n,\chi_p) = \sum_{\substack{k=1\\ 2\nmid k,\, p\nmid k}}^{\infty} \frac{1}{k}\Big\{ \qquad \Big\} F(n,k) + \sum_{\substack{k=1\\ 2\mid k,\, p\nmid k}}^{\infty} \frac{1}{k}\Big\{ \qquad \Big\} F(n,k)$$

$$+ \sum_{\substack{k=1\\ 4\mid k,\, p\nmid k}}^{\infty} \frac{1}{k}\Big\{ \qquad \Big\} F(n,k)$$

$$+ \sum_{\substack{k=1\\ 2\nmid k,\, p\mid k}}^{\infty} \sum_{m=0}^{2} \frac{1}{k}\Big\{ \qquad \Big\} G(n,m,k).$$

- F and G grow like $\exp(C\sqrt{n}/k)$ and $\exp(C'\sqrt{(8-3m)n}/k)$, resp..

The big idea (for $p = 17$)

$$\mathfrak{p}(n,\chi_p) = \sum_{\substack{k=1 \\ 2\nmid k,\, p\nmid k}}^{\infty} \frac{1}{k}\Big\{\lambda_k \mathfrak{L}_k(n)\Big\} F(n,k) + \sum_{\substack{k=1 \\ 2\mid k,\, p\nmid k}}^{\infty} \frac{1}{k}\Big\{\lambda_k \mathfrak{L}_k(n)\Big\} F(n,k)$$
$$+ \sum_{\substack{k=1 \\ 4\mid k,\, p\nmid k}}^{\infty} \frac{1}{k}\Big\{\lambda_k \mathfrak{L}_k(n)\Big\} F(n,k)$$
$$+ \sum_{\substack{k=1 \\ 2\nmid k,\, p\mid k}}^{\infty} \sum_{m=0}^{2} \frac{1}{k}\Big\{c_m \mathfrak{L}_k^{+}(n,m)\Big\} G(n,m,k).$$

- F and G grow like $\exp(C\sqrt{n}/k)$ and $\exp(C'\sqrt{(8-3m)n}/k)$, resp..
- The λ_k are algebraic numbers, related to "cyclotomy". E.g.,

$$\lambda_1 = \lambda_2 = \sqrt{1 + \tfrac{4}{\sqrt{17}}}.$$

The big idea (for $p = 17$)

$$\mathfrak{p}(n, \chi_p) = \sum_{\substack{k=1 \\ 2\nmid k,\, p\nmid k}}^{\infty} \frac{1}{k}\Big\{\lambda_k \mathfrak{L}_k(n)\Big\} F(n,k) + \sum_{\substack{k=1 \\ 2|k,\, p\nmid k}}^{\infty} \frac{1}{k}\Big\{\lambda_k \mathfrak{L}_k(n)\Big\} F(n,k)$$

$$+ \sum_{\substack{k=1 \\ 4|k,\, p\nmid k}}^{\infty} \frac{1}{k}\Big\{\lambda_k \mathfrak{L}_k(n)\Big\} F(n,k)$$

$$+ \sum_{\substack{k=1 \\ 2\nmid k,\, p|k}}^{\infty} \sum_{m=0}^{2} \frac{1}{k}\Big\{c_m \mathfrak{L}_k^{+}(n,m)\Big\} G(n,m,k).$$

- F and G grow like $\exp(C\sqrt{n}/k)$ and $\exp(C'\sqrt{(8-3m)n}/k)$, resp..
- The λ_k are algebraic numbers, related to "cyclotomy".
- The $\mathfrak{L}_k$ and $\mathfrak{L}_k^{+}$ are essentially character-twisted incomplete Kloosterman sums.

The big idea (for $p = 17$)

$$\mathfrak{p}(n,\chi_p) = \sum_{\substack{k=1 \\ 2\nmid k,\, p\nmid k}}^{\infty} \frac{1}{k}\Big\{\lambda_k \mathfrak{L}_k(n)\Big\} F(n,k) + \sum_{\substack{k=1 \\ 2|k,\, p\nmid k}}^{\infty} \frac{1}{k}\Big\{\lambda_k \mathfrak{L}_k(n)\Big\} F(n,k)$$

$$+ \sum_{\substack{k=1 \\ 4|k,\, p\nmid k}}^{\infty} \frac{1}{k}\Big\{\lambda_k \mathfrak{L}_k(n)\Big\} F(n,k)$$

$$+ \sum_{\substack{k=1 \\ 2\nmid k,\, p|k}}^{\infty} \sum_{m=0}^{2} \frac{1}{k}\Big\{c_m \mathfrak{L}_k^{+}(n,m)\Big\} G(n,m,k).$$

- F and G grow like $\exp(C\sqrt{n}/k)$ and $\exp(C'\sqrt{(8-3m)n}/k)$, resp..
- The λ_k are algebraic numbers, related to "cyclotomy".
- The $\mathfrak{L}_k$ and $\mathfrak{L}_k^{+}$ are essentially character-twisted incomplete Kloosterman sums.
- $c_0 = c_2 = 1$ and $c_1 = 0$ (Key!).

The big idea for $p = 17$ (cont.)

Because

$$(k, 2p) = 1 \Longrightarrow F(n, 2k) = 2F(n, k),$$

we condense the first two sums:

$$\begin{aligned}
\mathfrak{p}(n, \chi_p) = & \sum_{\substack{k=1 \\ 2\nmid k,\, p\nmid k}}^{\infty} \frac{1}{k}\Big\{\lambda_k \mathfrak{L}_k(n) + \lambda_{2k}\mathfrak{L}_{2k}(n)\Big\} F(n, k) \\
& + \sum_{\substack{k=1 \\ 4|k,\, p\nmid k}}^{\infty} \frac{1}{k}\Big\{\lambda_k \mathfrak{L}_k(n)\Big\} F(n, k) \\
& + \sum_{\substack{k=1 \\ 2\nmid k,\, p|k}}^{\infty} \sum_{m=0}^{2} \frac{1}{k}\Big\{c_m \mathfrak{L}_k^{+}(n, m)\Big\} G(n, m, k).
\end{aligned}$$

Examining $\mathfrak{L}_k$ and $\mathfrak{L}_k^+$; examples

For some small k,

$$\begin{aligned}
\mathfrak{L}_1(n) &= 1,\\
\mathfrak{L}_2(n) &= (-1)^n,\\
\mathfrak{L}_3(n) &= 2\cos\big(\tfrac{2\pi}{3}(n-\tfrac{2}{3})\big),\\
\mathfrak{L}_4(n) &= 2\cos\big(\tfrac{2\pi}{4}(n-2)\big),\\
\mathfrak{L}_5(n) &= 2\cos(\tfrac{2\pi}{5}(n+1)) + 2\cos(\tfrac{2\pi}{5}(2n-1)),\\
&\ \ \vdots\\
\mathfrak{L}_{17}^+(n) &= 2\cos(\tfrac{2\pi}{17}(n-\tfrac{7}{2})) + 2\cos(\tfrac{2\pi}{17}(2n-5))\\
&\qquad + 2\cos(\tfrac{2\pi}{17}(4n-\tfrac{1}{2})) + 2\cos(\tfrac{2\pi}{17}(8n+8)).
\end{aligned}$$

Examining $\mathfrak{L}_k$ and $\mathfrak{L}_k^+$; the χ-twisted Dedekind sums

Definition (Classical Dedekind sums)

$$s(h,k) := \sum_{\mu \bmod k} \left(\left(\frac{h\mu}{k}\right)\right)\left(\left(\frac{\mu}{k}\right)\right),$$

with

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & x \notin \mathbb{Z}, \\ 0 & x \in \mathbb{Z}. \end{cases}$$

Examining $\mathfrak{L}_k$ and $\mathfrak{L}_k^+$; the χ-twisted Dedekind sums

Definition (Classical Dedekind sums)

$$s(h,k) := \sum_{\mu \bmod k} \left(\left(\frac{h\mu}{k}\right)\right)\left(\left(\frac{\mu}{k}\right)\right),$$

with

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & x \notin \mathbb{Z}, \\ 0 & x \in \mathbb{Z}. \end{cases}$$

$$s_\chi(h,k) := \sum_{\mu \bmod [k,p]} \chi(\mu)\left(\left(\frac{h\mu}{k}\right)\right)\left(\left(\frac{\mu}{[k,p]}\right)\right),$$

where $[k,p] = \mathrm{lcm}(k,p)$.

Examining $\mathfrak{L}_k$ and $\mathfrak{L}_k^+$; the χ-twisted Dedekind sums

Definition (Classical Dedekind sums)

$$s(h,k) := \sum_{\mu \bmod k} \left(\!\left(\frac{h\mu}{k}\right)\!\right)\left(\!\left(\frac{\mu}{k}\right)\!\right),$$

with

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & x \notin \mathbb{Z}, \\ 0 & x \in \mathbb{Z}. \end{cases}$$

$$s_\chi(h,k) := \sum_{\mu \bmod [k,p]} \chi(\mu)\left(\!\left(\frac{h\mu}{k}\right)\!\right)\left(\!\left(\frac{\mu}{[k,p]}\right)\!\right),$$

where $[k,p] = \mathrm{lcm}(k,p)$.

Remark

Sums $s_\chi(h,k)$ (and more general) were introduced and studied by Berndt in connection with transformations of $\log \eta(\tau)$.

The Λ sums

Definition

For k with $2 \nmid k$ and $p \mid k$, define

$$\mathfrak{L}_k^+(n) := \mathfrak{L}_k^+(n,0) := \sideset{}{'}\sum_{\substack{h \pmod{k} \\ \chi(h)=1}} \exp\left\{\pi i \Lambda(h,k) - 2\pi i h n / k\right\}.$$

The Λ sums

Definition

For k with $2 \nmid k$ and $p \mid k$, define

$$\mathfrak{L}_k^+(n) := \mathfrak{L}_k^+(n,0) := \sideset{}{'}\sum_{\substack{h \pmod k \\ \chi(h)=1}} \exp\{\pi i \Lambda(h,k) - 2\pi i hn/k\}.$$

Definition (cont.)

here

$$\Lambda(h,k) := \sum_{\substack{\mu \bmod k \\ \chi(\mu)=1}} \chi(\mu)\left(\left(\frac{h\mu}{k}\right)\right)\left(\left(\frac{\mu}{k}\right)\right) + \sum_{\substack{\mu \bmod k \\ \chi(\mu)=-1}} \chi(\mu)\left\{\left(\left(\frac{2h\mu}{k}\right)\right) - \left(\left(\frac{h\mu}{k}\right)\right)\right\}\left(\left(\frac{\mu}{k}\right)\right).$$

The Λ sums

Definition

For k with $2 \nmid k$ and $p \mid k$, define

$$\mathfrak{L}_k^+(n) := \mathfrak{L}_k^+(n,0) := \sum_{\substack{h \,(\mathrm{mod}\, k) \\ \chi(h)=1}}{}^{'} \exp\{\pi i \Lambda(h,k) - 2\pi i h n/k\}.$$

A simpler formula for $\Lambda(h,k)$

$$\Lambda(h,k) = \tfrac{1}{2}\{s_\chi(h,k) - s_\chi(2h,k)\} + \tfrac{1}{2}\{s(2h,k) - s(2hp,k)\}.$$

“Kloosterman” sums

For simplicity: Assume $p \equiv 1 \pmod{8}$, $p \mid k$, $2 \nmid k$, and $3 \nmid k$.

“Kloosterman” sums

For simplicity: Assume $p \equiv 1 \pmod 8$, $p \mid k$, $2 \nmid k$, and $3 \nmid k$. Then

$$\mathfrak{L}_k^+(n) := \sideset{}{'}\sum_{\substack{h \pmod k \\ \chi(h)=1}} \exp\left\{\pi i \Lambda(h,k) - 2\pi i h n/k\right\}$$

$$= \sideset{}{'}\sum_{\substack{h \pmod k \\ \chi(h)=1}} \exp\left\{\left[24k\Lambda(h,k) - 48hn\right]\frac{\pi i}{24k}\right\},$$

“Kloosterman” sums

For simplicity: Assume $p \equiv 1 \pmod 8$, $p \mid k$, $2 \nmid k$, and $3 \nmid k$. Then

$$\begin{aligned}\mathfrak{L}_k^+(n) &:= \sideset{}{'}\sum_{\substack{h \pmod k\\ \chi(h)=1}} \exp\left\{\pi i \Lambda(h,k) - 2\pi i h n/k\right\}\\ &= \sideset{}{'}\sum_{\substack{h \pmod k\\ \chi(h)=1}} \exp\left\{\left[24k\Lambda(h,k) - 48hn\right]\frac{\pi i}{24k}\right\},\end{aligned}$$

$$24k\Lambda(h,k) \equiv 9\overline{h}\chi(h)B_2(\chi) - (p-1)(2h + \overline{2}\overline{h}) \pmod k,$$

where $h\overline{h} \equiv 1 \pmod k$ and $B_2(\chi) := \frac{1}{p}\sum_{\mu=1}^{p-1} \mu^2\chi(\mu)$;

"Kloosterman" sums

For simplicity: Assume $p \equiv 1 \pmod 8$, $p \mid k$, $2 \nmid k$, and $3 \nmid k$. Then

$$\mathfrak{L}_k^+(n) := \sideset{}{'}\sum_{\substack{h \pmod k \\ \chi(h)=1}} \exp\{\pi i \Lambda(h,k) - 2\pi i h n/k\}$$

$$= \sideset{}{'}\sum_{\substack{h \pmod k \\ \chi(h)=1}} \exp\left\{ \left[24k\Lambda(h,k) - 48hn \right] \frac{\pi i}{24k} \right\},$$

$$24k\Lambda(h,k) \equiv 9\overline{h}\chi(h)B_2(\chi) - (p-1)(2h + \overline{2}\overline{h}) \pmod k,$$

where $h\overline{h} \equiv 1 \pmod k$ and $B_2(\chi) := \frac{1}{p}\sum_{\mu=1}^{p-1} \mu^2 \chi(\mu)$;

$$24k\Lambda(h,k) \equiv 0 \pmod 3;$$

"Kloosterman" sums

For simplicity: Assume $p \equiv 1 \pmod 8$, $p \mid k$, $2 \nmid k$, and $3 \nmid k$. Then

$$\mathfrak{L}_k^+(n) := \sideset{}{'}\sum_{\substack{h \pmod k \\ \chi(h)=1}} \exp\{\pi i \Lambda(h,k) - 2\pi i h n/k\}$$

$$= \sideset{}{'}\sum_{\substack{h \pmod k \\ \chi(h)=1}} \exp\left\{ \left[24k\Lambda(h,k) - 48hn \right] \frac{\pi i}{24k} \right\},$$

$$24k\Lambda(h,k) \equiv 9\bar{h}\chi(h)B_2(\chi) - (p-1)(2h + \bar{2}\bar{h}) \pmod k,$$

where $h\bar{h} \equiv 1 \pmod k$ and $B_2(\chi) := \frac{1}{p}\sum_{\mu=1}^{p-1} \mu^2 \chi(\mu)$;

$$24k\Lambda(h,k) \equiv 0 \pmod 3;$$

also

$$24k\Lambda(h,k) \equiv \begin{cases} 8 \pmod{16} & h \text{ quartic } \pmod p \\ 0 \pmod{16} & h \text{ quadratic-nonquartic } \pmod p. \end{cases}$$

Ingredients for congruences of $24k\Lambda(h,k)$

Let $p|k$ and k odd. Define for $(h,k)=1$

$$\tau_k(h) := \#\{1 \leq \mu < k : \mu \text{ even},\ \chi(\mu) = -1, \text{ and } k\{h\mu/k\} \text{ odd.}\}$$

Ingredients for congruences of $24k\Lambda(h,k)$

Let $p|k$ and k odd. Define for $(h,k)=1$

$$\tau_k(h) := \#\{1 \le \mu < k : \mu \text{ even}, \chi(\mu) = -1, \text{ and } k\{h\mu/k\} \text{ odd.}\}$$

$$\tau_k(h) \equiv \sum_{\substack{1 \le \mu < k \\ \mu \text{ even}}} \left(\frac{1-\chi(\mu)}{2}\right)\left\{\frac{h\mu}{k}\right\} \pmod 2 \qquad (0 \le \{x\} < 1).$$

Ingredients for congruences of $24k\Lambda(h,k)$

Let $p|k$ and k odd. Define for $(h,k)=1$

$$\tau_k(h) := \#\{1 \le \mu < k : \mu \text{ even}, \chi(\mu) = -1, \text{ and } k\{h\mu/k\} \text{ odd.}\}$$

$$\tau_k(h) \equiv \sum_{\substack{1\le\mu<k \\ \mu \text{ even}}} \left(\frac{1-\chi(\mu)}{2}\right)\left\{\frac{h\mu}{k}\right\} \pmod 2 \qquad (0 \le \{x\} < 1).$$

When $\chi(h) = 1$, one has

$$24k\Lambda(h,k) \equiv 8\tau_k(2h) \pmod{16},$$

so knowing $\tau_k(h) \pmod 2$ will determine $24k\Lambda(h,k) \pmod{16}$.

Ingredients for congruences of $24k\Lambda(h,k)$

Let $p|k$ and k odd. Define for $(h,k)=1$

$$\tau_k(h) := \#\{1 \le \mu < k : \mu \text{ even}, \chi(\mu) = -1, \text{ and } k\{h\mu/k\} \text{ odd.}\}$$

$$\tau_k(h) \equiv \sum_{\substack{1\le\mu<k \\ \mu \text{ even}}} \left(\frac{1-\chi(\mu)}{2}\right)\left\{\frac{h\mu}{k}\right\} \pmod 2 \qquad (0 \le \{x\} < 1).$$

When $\chi(h)=1$, one has

$$24k\Lambda(h,k) \equiv 8\tau_k(2h) \pmod{16},$$

so knowing $\tau_k(h) \pmod 2$ will determine $24k\Lambda(h,k) \pmod{16}$.

Lemma

Suppose that $p|k$ and k odd, and that $(h,k)=1$. Then

$$\tau_k(h) \equiv \begin{cases} 1 \pmod 2 & h \text{ quartic} \pmod p, \\ 0 \pmod 2 & h \text{ quadratic-nonquartic} \pmod p. \end{cases}$$

Other interesting congruences; reciprocity

The following reciprocity laws are due to Berndt:

1. For $p|k$ with $(h,k)=1$ we have
$$s_\chi(h,k)+\chi(h)s_\chi(\bar{k},h)=\frac{h^2+\chi(h)}{2hk}B_2(\chi)\qquad(\bar{k}k\equiv 1\pmod{h}).$$
2. When $p\nmid k$ one has
$$s_\chi(h,k)+\tilde{s}_\chi(h,k)=\frac{h^2+1}{2hk}B_2(\chi),$$
where
$$\tilde{s}_\chi(h,k):=\sum_{0\leqq j<h}S(j/h),$$
$$S(y):=\frac{1}{p}\sum_{0\leqq\mu<p}\sum_{0\leqq\nu<p}\mu\chi(\nu)\left[\frac{k\mu+ky+\nu}{p}\right].$$

Other interesting congruences; reciprocity

When $p \nmid k$ one has

$$s_\chi(h,k) + \tilde{s}_\chi(h,k) = \frac{h^2+1}{2hk} B_2(\chi),$$

where

$$\tilde{s}_\chi(h,k) := \sum_{0 \leqq j < h} S(j/h),$$

$$S(y) := \frac{1}{p} \sum_{0 \leqq \mu < p} \sum_{0 \leqq \nu < p} \mu \chi(\nu) \left[\frac{k\mu + ky + \nu}{p} \right].$$

Lemma

For $p \nmid k$, all k (even and odd), and $(h,k) = 1$, one has

$$\tilde{s}_\chi(h,k) \equiv \tfrac{1}{2}(\chi(k) - 1) \pmod 2.$$

When $(k, p) = 1$

$$\mathfrak{L}_k(n) := \sum_{h \pmod{k}}{}' \exp\{\pi i \Lambda(h,k) - 2\pi i h n/k\}$$

Lemma

Let $p \neq 5$ and $p \equiv 1 \pmod 4$. For odd k coprime to p, one has

$$\mathfrak{L}_{2k}(n) = \mathfrak{L}_2(n)\mathfrak{L}_k(n) = (-1)^n \mathfrak{L}_k(n).$$

Moreover, when $p \equiv 1 \pmod 8$ one has $\lambda_k = \lambda_{2k}$ for all k with $(k, 4p) = 1$. Thus, when $p \equiv 1 \pmod 8$, the first sum in (14) vanishes for all odd n.

When $(k, p) = 1$

$$\mathfrak{L}_k(n) := \sum_{h \pmod{k}}{}' \exp\{\pi i \Lambda(h,k) - 2\pi i h n / k\}$$

Lemma

Let $p \neq 5$ and $p \equiv 1 \pmod 4$. For odd k coprime to p, one has

$$\mathfrak{L}_{2k}(n) = \mathfrak{L}_2(n)\mathfrak{L}_k(n) = (-1)^n \mathfrak{L}_k(n).$$

Moreover, when $p \equiv 1 \pmod 8$ one has $\lambda_k = \lambda_{2k}$ for all k with $(k, 4p) = 1$. Thus, when $p \equiv 1 \pmod 8$, the first sum in (14) vanishes for all odd n.

Lemma

For $p \equiv 1 \pmod 8$ and $(k, p) = 1$, one has $\mathfrak{L}_{4k}(n) = 0$ for odd n. Thus, the second sum in (14) vanishes for all odd n.

The sums $\mathfrak{L}_k^+$ for $p \mid k$, with $p = 17$.

Recall/Define

Let $2 \nmid k$ and $p \mid k$, and set

$$\mathfrak{L}_k^+(n,m) := \sum_{\substack{h \pmod k \\ \chi(h)=+1}}{}' \exp\left\{\pi i \Lambda(h,k) - 2\pi i(hn + \bar{2}\bar{h}m)/k\right\},$$

where again

$$\Lambda(h,k) = \tfrac{1}{2}\left\{s_\chi(h,k) - s_\chi(2h,k)\right\} + \tfrac{1}{2}\left\{s(2h,k) - s(2hp,k)\right\}.$$

The sums $\mathfrak{L}_k^+$ for $p \mid k$, with $p = 17$.

Recall/Define

Let $2 \nmid k$ and $p \mid k$, and set

$$\mathfrak{L}_k^+(n,m) := \sum_{\substack{h \pmod{k} \\ \chi(h)=+1}}' \exp\{\pi i \Lambda(h,k) - 2\pi i(hn + \bar{2}\bar{h}m)/k\},$$

where again

$$\Lambda(h,k) = \tfrac{1}{2}\{s_\chi(h,k) - s_\chi(2h,k)\} + \tfrac{1}{2}\{s(2h,k) - s(2hp,k)\}.$$

Idea

For properly "aligned" n and m, the exp-terms in $\mathfrak{L}_k^+(n,m)$ for quartic h cancel the exp-terms with quadratic-nonquartic h.

Serious "luck"

For $p = 17$, $p \mid k$, we have

$$\begin{aligned} \mathfrak{L}_k^+(n,0) &= 0 \qquad \text{for } n \equiv 0, 2, 8, 10 \pmod{17}, \\ \mathfrak{L}_k^+(n,1) &= 0 \qquad \text{for } n \equiv 1, 4, 6, 9 \pmod{17}, \\ \mathfrak{L}_k^+(n,2) &= 0 \qquad \text{for } n \equiv 0, 2, 8, 10 \pmod{17}. \end{aligned}$$

Serious "luck"

For $p = 17$, $p \,|\, k$, we have

$$\begin{aligned} \mathfrak{L}_k^+(n,0) &= 0 && \text{for } n \equiv 0,2,8,10 \pmod{17}, \\ \mathfrak{L}_k^+(n,1) &= 0 && \text{for } n \equiv 1,4,6,9 \pmod{17}, \\ \mathfrak{L}_k^+(n,2) &= 0 && \text{for } n \equiv 0,2,8,10 \pmod{17}. \end{aligned}$$

Generally

$$\mathfrak{L}_k^+(n,m_1), \quad \mathfrak{L}_k^+(n,m_2) \quad \text{won't simultaneously vanish if } m_1 \neq m_2.$$

Serious "luck"

For $p = 17$, $p \mid k$, we have

$$\begin{aligned} \mathfrak{L}_k^+(n,0) &= 0 \qquad \text{for } n \equiv 0,2,8,10 \pmod{17}, \\ \mathfrak{L}_k^+(n,1) &= 0 \qquad \text{for } n \equiv 1,4,6,9 \pmod{17}, \\ \mathfrak{L}_k^+(n,2) &= 0 \qquad \text{for } n \equiv 0,2,8,10 \pmod{17}. \end{aligned}$$

Generally

$$\mathfrak{L}_k^+(n,m_1), \quad \mathfrak{L}_k^+(n,m_2) \quad \text{won't simultaneously vanish if } m_1 \neq m_2.$$

For $p = 17$, coefficient of $G(n,m,k)$ is

$$\Big\{ c_0 \mathfrak{L}_k^+(n,0) + c_1 \mathfrak{L}_k^+(n,1) + c_2 \mathfrak{L}_k^+(n,2) \Big\},$$

but

Serious "luck"

For $p = 17$, $p \,|\, k$, we have

$$\begin{aligned} \mathfrak{L}_k^+(n,0) &= 0 \qquad \text{for } n \equiv 0, 2, 8, 10 \pmod{17}, \\ \mathfrak{L}_k^+(n,1) &= 0 \qquad \text{for } n \equiv 1, 4, 6, 9 \pmod{17}, \\ \mathfrak{L}_k^+(n,2) &= 0 \qquad \text{for } n \equiv 0, 2, 8, 10 \pmod{17}. \end{aligned}$$

Generally

$$\mathfrak{L}_k^+(n, m_1), \quad \mathfrak{L}_k^+(n, m_2) \quad \text{won't simultaneously vanish if } m_1 \neq m_2.$$

For $p = 17$, coefficient of $G(n, m, k)$ is

$$\Big\{ c_0 \mathfrak{L}_k^+(n,0) + c_1 \mathfrak{L}_k^+(n,1) + c_2 \mathfrak{L}_k^+(n,2) \Big\},$$

but

$$c_0 = c_2 = 1 \qquad c_1 = 0.$$

Are 5 and 17 special?

Definition

In describing the aformentioned vanishings, we may say that $\mathfrak{p}(n, \chi_5)$ and $\mathfrak{p}(n, \chi_{17})$ *vanish on some arithmetic progressions*.

Are 5 and 17 special?

Definition

In describing the aformentioned vanishings, we may say that $\mathfrak{p}(n, \chi_5)$ and $\mathfrak{p}(n, \chi_{17})$ *vanish on some arithmetic progressions*.

Theorem (D, 2024)

If $p \neq 2, 5$ and $p \not\equiv 1 \pmod 8$, then $\mathfrak{p}(n, \chi_p) \to \infty$ as $n \to \infty$. Thus, for these primes, the quantities $\mathfrak{p}(n, \chi_p)$ do not vanish on any arithmetic progressions.

Are 5 and 17 special?

Definition

In describing the aformentioned vanishings, we may say that $\mathfrak{p}(n, \chi_5)$ and $\mathfrak{p}(n, \chi_{17})$ *vanish on some arithmetic progressions*.

Theorem (D, 2024)

If $p \neq 2, 5$ and $p \not\equiv 1 \pmod 8$, then $\mathfrak{p}(n, \chi_p) \to \infty$ as $n \to \infty$. Thus, for these primes, the quantities $\mathfrak{p}(n, \chi_p)$ do not vanish on any arithmetic progressions.

Conjecture (Proof/work in-progress)

Among all primes $p > 2$, the quantities $\mathfrak{p}(n, \chi_p)$ and $\mathfrak{p}(n, \chi_p^{\dagger})$ vanish on certain arithmetic progressions only when $p = 5$ or $p = 17$.

Another interesting example

Definition

Fix $p = 13$ and $\chi = \chi_{13}$, and let

$$A(q) = \prod_{r=1}^{12} (\chi(r) q^r; q^{13})_\infty^{\chi(r)} = \frac{(q, q^3, q^4, q^9, q^{10}, q^{12}; q^{13})_\infty}{(-q^2, -q^5, -q^6, -q^7, -q^8, -q^{11}; q^{13})_\infty}.$$

Define the coefficients a_n via

$$A(q) = \sum_{n=0}^{\infty} a_n q^n.$$

Theorem (D, McLaughlin, 2025+)

One has

$$a_{13m+3} = a_{13m+9} = a_{13m+11} = 0 \qquad \textit{for } m \geq 0$$

My thanks for attending!