Partitions and A Multi-dimensional Continued Fraction Algorithm

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Goal

Use the dynamics of the triangle map (a type of multi-dimensional continued fraction algorithm) to create an almost internal symmetry on the space of all partitions of a given integer N.

New Partition Identities

As one type of example, we can show

Theorem

Every number has as many integer partitions into partitions with $\lambda_1 < \lambda_2 + \lambda_m$ *as into partitions with* $k_1 > k_m$*. Similarly, every number has as many integer partitions into partitions with* $\lambda_1 > \lambda_2 + \lambda_m$ *as into partitions with* $k_1 < k_m$*.*

Here a partition

$$(\lambda_1^{k_1}, \ldots, \lambda_m^{k_m})$$

is written as

$$(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m]$$

New Partition Identities

We will show how it is easy to generate such identities.

The triangle map

1. Periodicity and algebraic numbers
2. How to divide many small numbers into a big one

The Triangle Map

Roots of Multi-dimensional Continued Fractions:

1. Generalize the fact that a number has an eventually periodic continued fraction expansion if and only if it is a quadratic irrational.
2. Finding best Diophantine approximations of n-tuples of reals by n-tuples of rationals
3. As a rich source of dynamical systems, starting with Gauss on continued fractions all the way to the current work on interval exchange maps.

The Triangle Map

Let α be a real number.

1. α is a rational if and only if its decimal expansion is eventually periodic.
2. α is a quadratic irrational if and only if its continued fraction expansion is eventually periodic.

$$\sqrt{2} = 1 + \cfrac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}$$

The Triangle Map

Hermite Problem:

Find a way to represent any real number α as a sequence of integers so that the sequence is eventually periodic if and only if α is a cubic irrational.

The Triangle Map

How to divide little number into big number?

Really only one way: Euclidean algorithm

The Triangle Map

Take

$$\lambda_1 = 23, \lambda_2 = 7.$$

We have

$$\begin{array}{rcl} 23 & = & 3 \cdot 7 + 2 \\ 7 & = & 3 \cdot 2 + 1 \\ 2 & = & 2 \cdot 1 + 0 \end{array}$$

$$\frac{23}{7} = 3 + \frac{1}{3 + \frac{1}{2}}.$$

The Triangle Map

Euclidean algorithm can be written as

$$\begin{aligned}(23,7) &\rightarrow (7, 23 - 3 \cdot 7)\\ &= (7,2)\\ &\rightarrow (2, 7 - 3 \cdot 2)\\ &= (2,1)\\ &\rightarrow (1, 2 - 2 \cdot 1)\\ &= (1,0)\end{aligned}$$

The Triangle Map

There is the slow (additive, Farey) version

$$
\begin{aligned}
(23,7) &\rightarrow (23-7,7)=(16,7)\\
&\rightarrow (16-7,7)=(9,7)\\
&\rightarrow (7,9-7)=(7,2)\\
&\rightarrow (7-2,2)=(5,2))\\
&\rightarrow (5-2,2)=(3,2)\\
&\rightarrow (2,3-2)=(2,1)\\
&\rightarrow (2-1,1)=(1,1)\\
&\rightarrow (1,1-1)=(1,0)
\end{aligned}
$$

The Triangle Map

There is the slow (additive, Farey) version.
For $\lambda_1 > \lambda_2 > 0$, we have

$$(\lambda_1, \lambda_2) \xrightarrow{T_0} (\lambda_2, \lambda_1 - \lambda_2)$$

if

$$\lambda_2 > \lambda_1 - \lambda_2$$

and

$$(\lambda_1, \lambda_2) \xrightarrow{T_1} (\lambda_1 - \lambda_2, \lambda_2)$$

if

$$\lambda_2 < \lambda_1 - \lambda_2$$

The Triangle Map

Rephrased by two elements of $SL(2,\mathbb{Z})$:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_2 \\ \lambda_1 - \lambda_2 \end{pmatrix}$$

and

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 - \lambda_2 \\ \lambda_2 \end{pmatrix}$$

Set

$$T_0 = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, T_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

The Triangle Map

Given three numbers

$$\lambda_1 > \lambda_2 > \lambda_3 > 0$$

or, more generally, n numbers

$$\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0$$

There are many (too many) ways to divide little numbers into the big λ_1.

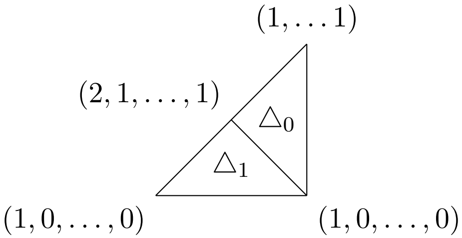
The Triangle Map

Domain:

$$\triangle = \{\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0\}$$

Subdomains

$$\begin{aligned} \triangle_0 &= \{\lambda_1 < \lambda_2 + \lambda_m\} \\ \triangle_1 &= \{\lambda_1 > \lambda_2 + \lambda_m\} \end{aligned}$$



The Triangle Map

The Slow-Triangle map $T : \triangle_0 \cup \triangle_1 \to \triangle$ by

$$\begin{aligned} T(\lambda_1, \ldots, \lambda_m) &= \begin{cases} T_0(\lambda_1, \ldots, \lambda_m), & \text{if } \lambda_2 + \lambda_m > \lambda_1 \\ T_1(\lambda_1, \ldots, \lambda_m), & \text{if } \lambda_2 + \lambda_m < \lambda_1 \end{cases} \\ &= \begin{cases} (\lambda_2, \lambda_3, \ldots, \lambda_m, \lambda_1 - \lambda_2), & \text{if } \lambda_2 + \lambda_m > \lambda_1 \\ (\lambda_1 - \lambda_m, \lambda_2, \ldots, \lambda_m), & \text{if } \lambda_2 + \lambda_m < \lambda_1 \end{cases} \end{aligned}$$

The Triangle Map

$$(7,5,1) \xrightarrow{T_1} (7-1,5,1) = (6,5,1)$$

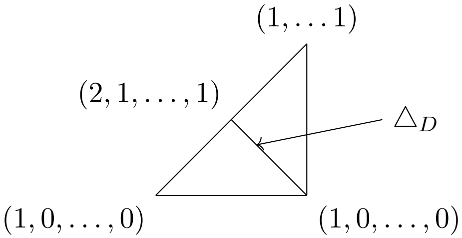
and

$$(7,5,4) \xrightarrow{T_0} (5,4,7-5) = (5,4,3)$$

The Triangle Map

For now, we ignore the diagonal

$$\triangle_D = \{(\lambda_1, \ldots, \lambda_m) \in \triangle : \lambda_2 + \lambda_m = \lambda_1\}$$



This is a set of measure zero, and hence ignored if concerned with dynamics.

The Triangle Map

A dynamical system on simplices.
Earlier work
TG (2001), S. Assaf, L. Chen, T. Cheslack-Postava, B. Cooper,
A. Diesl, TG, M. Lepinski and A. Schuyler (2005),
A. Messaoudi, A. Nogueira, and F. Schweiger (2009),
V. Berthé, W. Steiner and J. Thuswaldner (2021),
Fougeron and A. Skripchenko (2021),
C.Bonanno, A. Del Vigna and S. Munday (2021),
C. Bonanno and A. Del Vigna (2021),
H. Ito (2023) and some more

The Triangle Map

A point $\lambda \in \triangle$ has triangle sequence

$$(i_0, i_1, \ldots,)$$

where each $i_k = 0$ or 1 and

$$T^{(k)}(\lambda) \in \triangle_{i_k}$$

Theorem: If triangle sequence is eventually periodic, then

$$\frac{\lambda_2}{\lambda_1}, \ldots, \frac{\lambda_m}{\lambda_1}$$

are all in the same algebraic number field, of degree at most m.

The Triangle Map

$$T\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_m\end{pmatrix} = \begin{cases} T_0\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_m\end{pmatrix}, & \text{if } \lambda_2+\lambda_m>\lambda_1\\ T_1\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_m\end{pmatrix}, & \text{if } \lambda_2+\lambda_m<\lambda_1\end{cases}$$

The Triangle Map

where

$$T_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & -1 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad T_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Thus for $n = 3$, we have

$$T_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \quad \text{and} \quad T_1 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The Triangle Map

Summary: Start with cone

$$\triangle = \{\lambda_1 > \cdots > \lambda_m > 0\}$$

Split into two subcones

$$\begin{aligned} \triangle_0 &= \{\lambda_2 + \lambda_m > \lambda_1\} \\ \triangle_1 &= \{\lambda_2 + \lambda_m < \lambda_1\} \end{aligned}$$

Interate multiplication of the two $m \times m$ matrices

$$T_0 : \triangle_0 \to \triangle$$

$$T_1 : \triangle_0 \to \triangle$$

Integer Partitions

$p(n)$ is the number of ways of writing n as the sum of less than or equal t positive integers (ordering not mattering).
$p(7) = 15$ since

$$\begin{array}{ccc} 7 & 6+1 & 5+2 \\ 5+1+1 & 4+3 & 4+2+1 \\ 4+1+1+1 & 3+3+1 & 3+2+2 \\ 3+2+1+1 & 3+1+1+1+1 & 2+2+2+1 \\ 2+2+1+1+1 & 2+1+1+1+1+1 & 1+1+1+1+1+1+1. \end{array}$$

$$\begin{array}{cccc} (7) & (6,1) & (5,2) & (5,1^2) \\ (4,3) & (4,2,1) & (4,1^3) & (3^2,1) \\ (3,2^2) & (3,2,1^2) & (3,1^4) & (2^3,1) \\ (2^2,1^3) & (2,1^5) & (1^7). & \end{array}$$

Integer Partitions

$$\begin{array}{cccc} (7) & (6,1) & (5,2) & (5,1^2) \\ (4,3) & (4,2,1) & (4,1^3) & (3^2,1) \\ (3,2^2) & (3,2,1^2) & (3,1^4) & (2^3,1) \\ (2^2,1^3) & (2,1^5) & (1^7). & \end{array}$$

$$\begin{array}{cccc} (7)\times[1] & (6,1)\times[1,1] & (5,2)\times[1,1] & (5,1)\times[1,2] \\ (4,3)\times[1,1] & (4,2,1)\times[1,1,1] & (4,1)\times[1,3] & (3,1)\times[2,1] \\ (3,2)\times[1,2] & (3,2,1)\times[1,1,2] & (3,1)\times[1,4] & (2,1)\times[3,1] \\ (2,1)\times[2,3] & (2,1)\times[1,5] & (1)\times[7]. & \end{array}$$

Integer Partitions

$$\lambda = (\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] \vdash N$$

means

$$\begin{aligned} N &= k_1\lambda_1 + \ldots + k_m\lambda_m \\ &= \begin{pmatrix} k_1 & \cdots & k_m \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} \end{aligned}$$

Rhetoric: The λ_i's are the parts, the k_j's are the multiplicities

Triangle Map and Integer Partitions

$$\mathcal{P}(N) = \{\text{partition space for } N\}$$

For

$$\overrightarrow{\lambda} \times \overrightarrow{k} \in \mathcal{P}(N)$$

we have

$$\overrightarrow{\lambda} \in \triangle.$$

Apply the triangle map T.

Triangle Map and Integer Partitions

$$\begin{aligned} N &= k_1\lambda_1 + \ldots + k_m\lambda_m \\ &= \begin{pmatrix} k_1 & \cdots & k_m \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} \\ &= \begin{pmatrix} k_1 & \cdots & k_m \end{pmatrix} T_i^{-1} \cdot T_i \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} \end{aligned}$$

Triangle Map and Integer Partitions

$$(\lambda_1, \dots, \lambda_m) \times [k_1, \dots, k_m]$$
$$\tilde{T}_0 \downarrow$$
$$(\lambda_2, \lambda_3 \dots, \lambda_m, \lambda_1 - \lambda_2) \times [k_1 + k_2, k_3, \dots, k_m, k_1]$$

if $\lambda_2 + \lambda_m > \lambda_1$ and

$$(\lambda_1, \dots, \lambda_m) \times [k_1, \dots, k_m]$$
$$\tilde{T}_1 \downarrow$$
$$(\lambda_1 - \lambda_m, \lambda_2 \dots, \lambda_m,) \times [k_1, \dots, k_{m-1}, k_1 + k_m]$$

if $\lambda_2 + \lambda_m < \lambda_1$

Triangle Map and Integer Partitions

For this to “work”, we need both T_0^{-1} and T_1^{-1} to have non-negative entries. This is true for the triangle map. It is false for most (but not all) other multi-dimensional continued fraction algorithms.

Very few multi-dimensional continued fraction algorithms are *partition friendly*.

Why?

Why are a few types of division *partition friendly*, and why are most not?

Triangle Map and Integer Partitions

$$\begin{aligned}(14,7,6,5)\times[1,0,0,0] &\xrightarrow{\tilde{T}_1} (9,7,6,5)\times[1,0,0,1]\\ &\xrightarrow{\tilde{T}_0} (7,6,5,2)\times[1,0,1,1]\\ &\xrightarrow{\tilde{T}_0} (6,5,2,1)\times[1,1,1,1]\end{aligned}$$

Since $6 = 5 + 1$ $((6,5,2,1) \in \triangle_D)$, for now we must stop

Triangle Map and Integer Partitions

$$\tilde{T} : \triangle - \triangle_D \to \triangle$$

is also *Young compatible.*

Triangle Map and Integer Partitions

To a given partition

$$(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m]$$

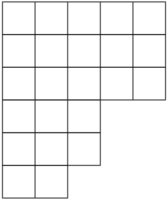
we associate the *Young shape*, a diagram $k_1 + \cdots + k_m$ rows such that there are k_1 rows with λ_1 squares on top of k_2 rows with λ_2 squares, and so on.

Triangle Map and Integer Partitions

For example, the Young shape for

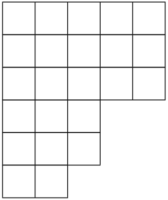
$$(5,3,2) \times [3,2,1] \vdash 23$$

is

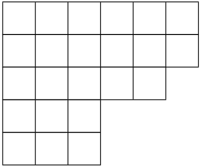


Triangle Map and Integer Partitions

Flip a Young shape, turning the rows into columns, to get the *conjugate partition*
Flipping the Young shape of the partition $(5, 3, 2) \times [3, 2, 1] \vdash 23$ of the previous example gives us the Young shape



$\sim_{\mathcal{C}}$



which represents the conjugate partition

$$(5, 3, 2) \times [3, 2, 1] \sim_{\mathcal{C}} (6, 5, 3) \times [2, 1, 21]$$

Triangle Map and Integer Partitions

$$(\lambda_1, \lambda_2) \times [k_1, k_2] \sim_{\mathcal{C}} (k_1 + k_2, k_1) \times [\lambda_2, \lambda_1 - \lambda_2]$$

and in general

$$\begin{gathered}(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] \\ \sim_{\mathcal{C}} \\ (k_1 + \ldots + k_m, k_1 + \ldots + k_{m-1}, \ldots, k_1) \\ \times \\ [\lambda_m, \lambda_{m-1} - \lambda_m, \ldots, \lambda_1 - \lambda_2]\end{gathered}$$

Triangle Map and Integer Partitions

Respects conjugation (is *Young compatible*):

Theorem

The diagram

$$\begin{array}{rcl} (\bar{\lambda}) \times [\bar{k}] & \sim_{\mathcal{C}} & \tilde{T}_0((\bar{\mu}) \times [\bar{l}]) \\ \tilde{T}_0 \downarrow & & \uparrow \tilde{T}_0 \\ \tilde{T}_0((\bar{\lambda} \times [\bar{k}])) & \sim_{\mathcal{C}} & (\bar{\mu}) \times [\bar{l}] \end{array}$$

when $\lambda_2 + \lambda_m > \lambda_1$ *and*

$$\begin{array}{rcl} (\bar{\lambda}) \times [\bar{k}] & \sim_{\mathcal{C}} & \tilde{T}_0 1((\bar{\mu}) \times [\bar{l}]) \\ \tilde{T}_1 \downarrow & & \uparrow \tilde{T}_1 \\ \tilde{T}_1((\bar{\lambda} \times [\bar{k}])) & \sim_{\mathcal{C}} & (\bar{\mu}) \times [\bar{l}] \end{array}$$

when $\lambda_2 + \lambda_m < \lambda_1$ *are both commutative.*

Triangle Map and Integer Partitions

It appears that the triangle map is the only multidimensional continued fraction algorithm that is both *partition friendly* and *Young Compatible*.

Matthew Phang conjectures that this is indeed true.

Triangle Map and Integer Partitions

What if

$$\lambda_1 = \lambda_2 + \lambda_m$$

Define

$$\begin{array}{c}(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] \\ \tilde{T}_D \downarrow \\ (\lambda_2, \lambda_3 \ldots, \lambda_m) \times [k_1 + k_2, k_3, \ldots, k_1 + k_m]\end{array}$$

Triangle Map and Integer Partitions

$$
\begin{aligned}
(14,7,6,5)\times[1,0,0,0] &\xrightarrow{\tilde{T}_1} (9,7,6,5)\times[1,0,0,1]\\
&\xrightarrow{\tilde{T}_0} (7,6,5,2)\times[1,0,1,1]\\
&\xrightarrow{\tilde{T}_0} (6,5,2,1)\times[1,1,1,1]\\
&\xrightarrow{\tilde{T}_D} (5,2,1)\times[2,1,2]\\
&\xrightarrow{\tilde{T}_1} (4,2,1)\times[2,1,4]\\
&\xrightarrow{\tilde{T}_1} (3,2,1)\times[2,1,6]\\
&\xrightarrow{\tilde{T}_D} (2,1)\times[3,8]
\end{aligned}
$$

Triangle Map and Integer Partitions

T_D, while weird in dynamics, is natural here.

$$(6,5,2,1) \times [1,1,1,1] \xrightarrow{\tilde{T_0}} (5,2,1,1) \times [2,1,1,1]$$

and

$$(6,5,2,1) \times [1,1,1,1] \xrightarrow{\tilde{T_1}} (5,5,2,1) \times [1,1,1,2]$$

If you concatenate, you get

$$(6,5,2,1) \times [1,1,1,1] \xrightarrow{\tilde{T_D}} (5,2,1) \times [2,1,2]$$

Triangle Map and Integer Partitions

$\mathcal{P}(N)$ = all partitions of N.

$$\begin{aligned}
\triangle &:= \{(\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m : \lambda_1 > \lambda_2 > \cdots > \lambda_m > 0\} \\
\triangle_0 &:= \{(\lambda_1, \ldots, \lambda_m) \in \triangle : \lambda_1 < \lambda_2 + \lambda_m\} \\
\triangle_1 &:= \{(\lambda_1, \ldots, \lambda_m) \in \triangle : \lambda_1 > \lambda_2 + \lambda_m\} \\
\triangle_D &:= \{(\lambda_1, \ldots, \lambda_m) \in \triangle : \lambda_1 = \lambda_2 + \lambda_m\}
\end{aligned}$$

Triangle Map and Integer Partitions

$\tilde{T}_0$ is one-to one on $\mathcal{P}(N) \cap \triangle_0$.

$\tilde{T}_1$ is one-to one on $\mathcal{P}(N) \cap \triangle_1$.

$\tilde{T}_D$ is not one-to one on $\mathcal{P}(N) \cap \triangle_D$.

Triangle Map and Integer Partitions

Idea:

1. Start with an interesting subset of $\mathcal{P}(N)$
2. Apply $\tilde{T}$
3. Count image

Triangle Map and Integer Partitions

Theorem

Every number has as many integer partitions into partitions with $\lambda_1 < \lambda_2 + \lambda_m$ as into partitions with $k_1 > k_m$. Similarly, every number has as many integer partitions into partitions with $\lambda_1 > \lambda_2 + \lambda_m$ as into partitions with $k_1 < k_m$.

Proof.

$$\begin{aligned} \mathrm{Im}(\tilde{T}_0(\triangle \cap \mathcal{P}(N)) &= \{k_1 > k_m\} \cap \mathcal{P}(N) \\ \mathrm{Im}(\tilde{T}_1(\triangle \cap \mathcal{P}(N)) &= \{k_1 < k_m\} \cap \mathcal{P}(N) \end{aligned}$$

□

Triangle Map and Integer Partitions

With

$$\begin{aligned}
\mathcal{O} &= \{(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] : \lambda_i \text{ odd}\} \\
\mathcal{F}_0 &= \{(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] : \lambda_m \text{ even}, \\
&\quad \lambda_i \text{ odd if } i < m, k_1 > k_m\} \\
\mathcal{F}_1 &= \{(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] : \lambda_1 \text{ even}, \\
&\quad \lambda_i \text{ odd if } i > 1, k_1 < k_m\}
\end{aligned}$$

then

$$p_{\mathcal{O}}(N) = (\text{number of odd factors of } N) + p_{\mathcal{F}_0}(N) + p_{\mathcal{F}_1}(N).$$

Triangle Map and Integer Partitions

There are many others.

They are easy to both create and to prove.

Triangle Map and Integer Partitions

It is also straightforward to write down corresponding generating functions.

Question: Starting with a proposed equality of generating functions, is there a direct proof (one not using the rhetoric of the triangle map)

Triangle Map and Integer Partitions

In dynamics, it is natural to study the cylinders of the map:

$$
\begin{aligned}
\triangle &= \{\lambda_1 > \cdots > \lambda_m > 0\} \\
\triangle_0 &= \{\lambda_2 + \lambda_m > \lambda_1\} \\
\triangle_1 &= \{\lambda_2 + \lambda_m < \lambda_1\} \\
\triangle_{00} &= \{\overrightarrow{\lambda} \in \triangle_0 : T_0(\overrightarrow{\lambda}) \in \triangle_0\} \\
\triangle_{01} &= \{\overrightarrow{\lambda} \in \triangle_0 : T_0(\overrightarrow{\lambda}) \in \triangle_1\} \\
\triangle_{10} &= \{\overrightarrow{\lambda} \in \triangle_1 : T_0(\overrightarrow{\lambda}) \in \triangle_0\} \\
\triangle_{11} &= \{\overrightarrow{\lambda} \in \triangle_1 : T_0(\overrightarrow{\lambda}) \in \triangle_1\} \\
&\vdots
\end{aligned}
$$

Triangle Map and Integer Partitions

For all m, we have $\lambda_1 > \cdots, > \lambda_m > 0$ and $k_i > 0$ for $i = 1, \ldots, m$.

sets	dim $= 2$	dim ≥ 3
$\triangle_0$	$2\lambda_2 > \lambda_1$	$\lambda_2 + \lambda_m > \lambda_1$
$\triangle_1$	$2\lambda_2 < \lambda_1$	$\lambda_2 + \lambda_m < \lambda_1$
$\triangle_D$	$2\lambda_2 = \lambda_1$	$\lambda_2 + \lambda_m = \lambda_1$
$\triangle_{00}$	$2\lambda_2 > \lambda_1, 2\lambda_1 > 3\lambda_2$	$\lambda_2 + \lambda_m > \lambda_1, 2\lambda_2 < \lambda_1 + \lambda_3$
$\triangle_{01}$	$2\lambda_2 > \lambda_1, 2\lambda_1 < 3\lambda_2$	$\lambda_2 + \lambda_m > \lambda_1, 2\lambda_2 > \lambda_1 + \lambda_3$
$\triangle_{10}$	$2\lambda_2 < \lambda_1, 3\lambda_2 > \lambda_1$	$\lambda_2 + \lambda_m < \lambda_1, \lambda_2 + 2\lambda_m > \lambda_1$
$\triangle_{11}$	$3\lambda_2 < \lambda_1$	$\lambda_2 + 2\lambda_m < \lambda_1$

These are natural subsets for dynamics (critical in proofs of ergodicity).

Triangle Map and Integer Partitions

Theorem

1. *Every number* N *has as many integer partitions into partitions with* $\lambda_1 < \lambda_2 + \lambda_m$ *and* $2\lambda_2 < \lambda_1 + \lambda_3$ *as into partitions with* $\lambda_1 < \lambda_2 + \lambda_m$ *and* $k_1 > k_m$*, i.e.*

$$p_{\triangle_{00}}(N) = p_{T_0(\triangle_{00})}(N).$$

2. *Every number has as many integer partitions into partitions with* $\lambda_1 < \lambda_2 + \lambda_m$ *and* $2\lambda_2 > \lambda_1 + \lambda_3$ *as into partitions with* $\lambda_1 > \lambda_2 + \lambda_m$ *and* $k_1 > k_m$*, i.e.*

$$p_{\triangle_{01}}(N) = p_{T_0(\triangle_{01})}(N).$$

Triangle Map and Integer Partitions

$$\begin{array}{c}(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] \\ \tilde{T}_0^{-1} \downarrow \\ (\lambda_1 + \lambda_m, \lambda_1, \ldots, \lambda_{m-1}) \times [k_m, k_1 - k_m . k_2, \ldots, k_{m-1}]\end{array}$$

if $k_1 > k_m$ and

$$\begin{array}{c}(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] \\ \tilde{T}_1^{-1} \downarrow \\ (\lambda_1 + \lambda_m, \lambda_2, \ldots, \lambda_m)) \times [k_1, k_2, \ldots, k_{m-1}, k_m - k_1]\end{array}$$

if $k_1 < k_m$ and

$$\begin{array}{c}(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] \\ \tilde{T}_D^{-1}(l) \downarrow \\ (\lambda_1 + \lambda_m, \lambda_1, \ldots, \lambda_m) \times [l, k_1 - l, k_2, \ldots, k_{m-1}, k_m - l]\end{array}$$

if $l < \min\{k_1, k_m\}$

Triangle Map and Integer Partitions

By looking at the inverse images of $(1) \times [N]$, get a tree structure on $\mathcal{P}(N)$.

- $(1) \times [7]$
  - $(2,1) \times [1,5]$
    - $(3,1) \times [1,4]$
      - $(4,1) \times [1,3]$
        - $(5,1) \times [1,2]$
          - $(6,1) \times [1,1]$
  - $(2,1) \times [2,3]$
  - $(2,1) \times [3,1]$
    - $(3,2) \times [1,2]$
      - $(5,2) \times [1,1]$
- $(7) \times [1]$

Triangle Map and Integer Partitions

A small part of the tree structure can be explained via the *natural extension*.

In dynamics, one can always change a many-to-one map to a one-to-one map (the natural extension). The word "natural" is here categorical.

This is what the map $\tilde{T}$ is doing. In the context of integer partitions, $\tilde{T}$ was forced upon us (which does suggest why it is categorically "natural" but also why it is natural in the colloquial sense of the word). Using Young conjugation, we get an interval symmetry on the natural extension. This is new.

Triangle Map and Integer Partitions

The natural extension of a map $T : X \to X$, where X is a measure space. (There are technical details floating about.) The map T can be many-to-one. For example, the triangle map acting on $\triangle$ is 2-to-1.

The natural extension is a map $S : Y \to Y$, where Y is a measure space, S is one-to-one, and there is an onto map

$$\pi : Y \to Y$$

making the following diagram commutative

$$\begin{array}{ccc} Y & \xrightarrow{S} & Y \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow{T} & X \end{array}$$

Triangle Map and Integer Partitions

Also need that if there is any other $U : Z \to Z$ with these properties, that it factors through $S : Y \to Y$.

Straightforward to prove a natural extension must exist, abstractly. Hard to make it concrete.

Triangle Map and Integer Partitions

For example, set-theoretically easy to find Y. It is simply the colimit of T.

Set

$$Y = \{(x_0, x_1, x_2, \ldots) : x_k \in X, T(x_{k+1}) = x_k\}$$

Then

$$S((x_0, x_1, x_2, \ldots) = ((T(x_0), x_0, x_1, \ldots)$$

and

$$\pi(x_0, x_1, x_2, \ldots) = x_0.$$

The work is showing that Y is a measure space with desired properties.

Triangle Map and Integer Partitions

We can find the natural extensions in a straightforward manner.

$$\mathcal{P}_m(N) = \text{all integer partitions of } N \text{ with } m \text{ parts} .$$

Elements are

$$(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m]$$

with all being non-negative integers and

$$\lambda_1 > \ldots > \lambda_m$$

with

$$k_1\lambda_1 + \ldots + k_m\lambda_m = N$$

Triangle Map and Integer Partitions

Why stick with integers? Allow real numbers.

$$\mathcal{P}_m(N) = \text{all partitions of } N \text{ with } m \text{ parts} \; .$$

Elements are

$$(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m]$$

with all being positive real numbers and

$$\lambda_1 > \ldots > \lambda_m$$

with

$$k_1\lambda_1 + \ldots + k_m\lambda_m = N$$

Triangle Map and Integer Partitions

Still have that $\tilde{T}$ will map

$$\mathcal{P}_m(N) - \{\lambda_1 = \lambda_2 + \lambda_m\} \rightarrow \mathcal{P}_m(N).$$

Triangle Map and Integer Partitions

Four natural subdomains:

$$
\begin{aligned}
\text{Domain}(\tilde{T}_0) &= \{\lambda_1 < \lambda_2 + \lambda_m\} \\
\text{Domain}(\tilde{T}_1) &= \{\lambda_1 > \lambda_2 + \lambda_m\} \\
\text{Image}(\tilde{T}_0) &= \{k_1 > k_m\} \\
\text{Image}(\tilde{T}_1) &= \{k_1 < k_m\}
\end{aligned}
$$

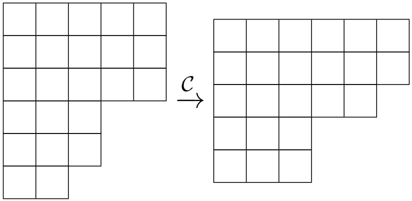
Triangle Map and Integer Partitions

Four natural subdomains:

$$\begin{aligned} A &= \text{Domain}(\tilde{T}_0) \cap \text{Image}(\tilde{T}_0) \\ B &= \text{Domain}(\tilde{T}_0) \cap \text{Image}(\tilde{T}_1) \\ C &= \text{Domain}(\tilde{T}_1) \cap \text{Image}(\tilde{T}_0) \\ D &= \text{Domain}(\tilde{T}_1) \cap \text{Image}(\tilde{T}_1) \end{aligned}$$

Triangle Map and Integer Partitions

Young conjugation is quite natural for integer partitions.



which represents the conjugate map

$$\mathcal{C}(5,3,2) \times [3,2,1] = (6,5,3) \times [2,1,21]$$

Triangle Map and Integer Partitions

Not sure what a Young shape would be for real numbers, but the map $\mathcal{C}$ still makes sense:

$$(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m]$$
$$\downarrow \mathcal{C}$$
$$(k_1 + \ldots + k_m, k_1 + \ldots + k_{m-1}, \ldots, k_1)$$
$$\times$$
$$[\lambda_m, \lambda_{m-1} - \lambda_m, \ldots, \lambda_1 - \lambda_2]$$

Triangle Map and Integer Partitions

$$\begin{array}{ccc}
\text{Domain}(\tilde{T}_0) \cap \text{Image}(\tilde{T}_0) & \xrightarrow{\mathcal{C}} & \text{Domain}(\tilde{T}_0) \cap \text{Image}(\tilde{T}_0) \\
\text{Domain}(\tilde{T}_0) \cap \text{Image}(\tilde{T}_1) & \xrightarrow{\mathcal{C}} & \text{Domain}(\tilde{T}_1) \cap \text{Image}(\tilde{T}_0) \\
\text{Domain}(\tilde{T}_1) \cap \text{Image}(\tilde{T}_0) & \xrightarrow{\mathcal{C}} & \text{Domain}(\tilde{T}_0) \cap \text{Image}(\tilde{T}_1) \\
\text{Domain}(\tilde{T}_1) \cap \text{Image}(\tilde{T}_1) & \xrightarrow{\mathcal{C}} & \text{Domain}(\tilde{T}_1) \cap \text{Image}(\tilde{T}_1)
\end{array}$$

Thus Young conjugation gives us an involution of the natural extension. This seems new.

Sources

1. "On integer partitions and continued fraction type algorithms" by Baalbaki, Bonanno, Del Vigna, TG,Isola in *Ramanujan J.* (2024)
2. "Generating new partition identities via a generalized continued fraction algorithm" by Baalbaki and TG in *Electron. J. Combinatorics* (2024),
3. "Ergodicity and Algebraticity of the Fast and Slow Triangle Maps" by TG and Lehmann Duke, https://arxiv.org/abs/2409.05822
4. "Methods for Obtaining Partition Identities Using the Selmer and Brun Algorithm", Phang, Senior Thesis,, Williams College, 2023.
5. "Tree Structures on Partitions Shaped by the Dynamics of the Triangle Map", Fox, Senior Thesis,, Williams College, 2024.

Questions

1. Is it true that the triangle map is the only multi-dimensional continued fraction algorithm that is both partition friendly and Young compatible?
2. Understand the nature of the tree structure
3. Direct proofs of generating function identities.
4. Find more identities
5. Can you put “q” into this language. (Maybe link with work of Sophie Morier-Genoud, Valentin Ovsienko and collaborators)
6. Use integer partitions to understand the dynamics
7. Multi-dimensional continued fractions can be linked to billards, translations surfaces, automata theory, etc. Can integer partition theory be used?

Homework

1. Find the path under $\tilde{T}$ of

$$(12, 7, 3, 2) \times [2, 3, 1, 5]$$

2. Find the tree structure for all $\mathcal{P}(N)$, for

$$N = 1, 2, 3, 4, 5, 6, 7, 8, 9.$$

3. For $\triangle = \{1 > \lambda_2 > \lambda_3 > 0\}$, find the values of λ_2, λ_3 whose triangle sequence is

$$(0, 1, 0, 1, 0, 1, \ldots).$$

THANKS