Hyperbinary partitions and q-deformed rationals

- Based on joint work with

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Outline

(I) Sequences

(II) Lattices

(III) Matrices

(I) Stern's diatomic sequence (or obfuscating sequence)

Defined by recurrence:

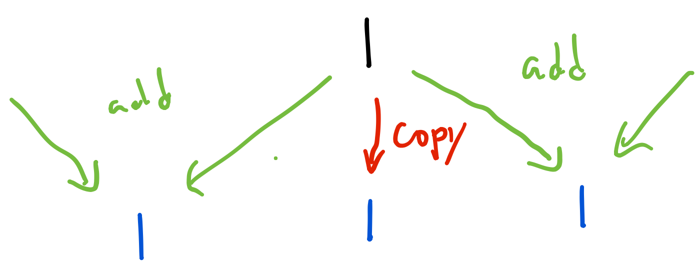
$$\mathrm{fusc}(0) = 0,$$
$$\mathrm{fusc}(1) = 1,$$

and for $n \geq 1$,

$$\mathrm{fusc}(2n) = \mathrm{fusc}(n),$$

$$\mathrm{fusc}(2n+1) = \mathrm{fusc}(n+1) + \mathrm{fusc}(n).$$

Alternatively, we can generate Stern's sequence from an array.



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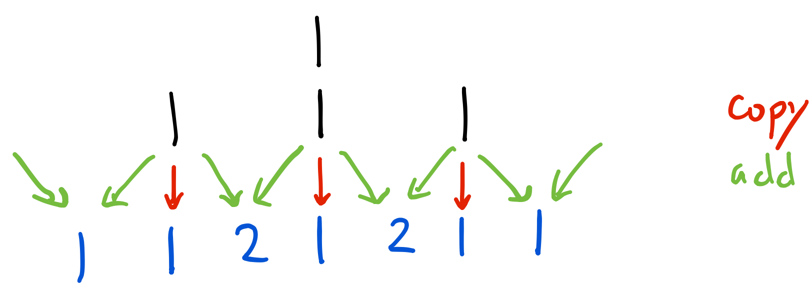

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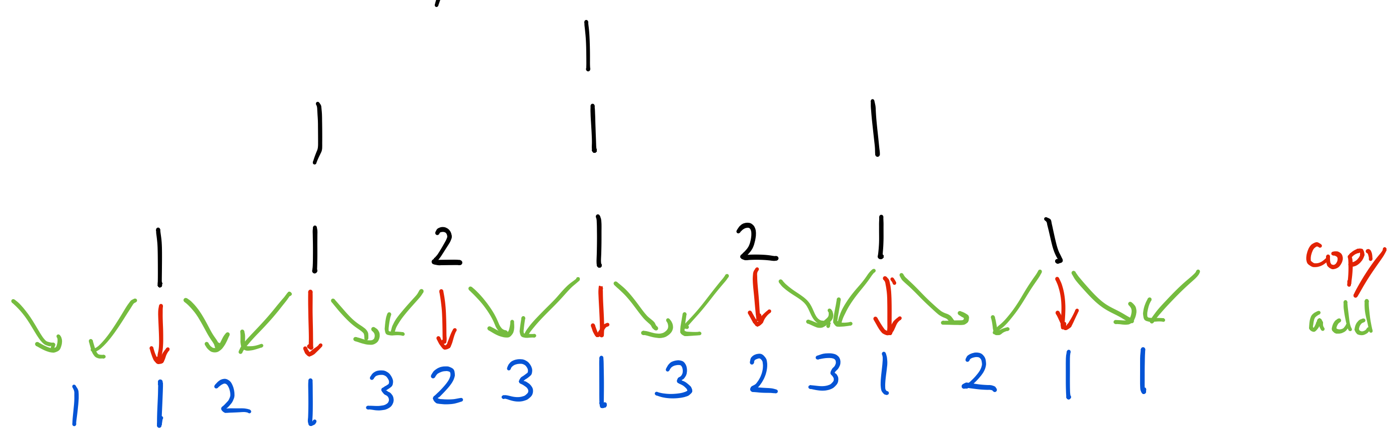

Stern's triangle

1

1 1 1

1 1 2 1 2 1 1

1 1 2 1 3 2 3 1 3 2 3 1 2 1 1

1 1 2 1 3 2 3 1 4 3 5 2 5 3 4 1 4 3 5 2 5 3 4 1 3 2 3 1 2 1 1

Notation: $\left\langle {n \atop k} \right\rangle = k^{th}$ term of n^{th} row of the triangle

Theorem [Stanley '19]

Let $$H_r(x) = \prod_{k=0}^{r-1}\left(1 + x^{2^k} + x^{2\cdot 2^k}\right),$$

and $$H(x) = \lim_{r\to\infty} H_r(x) = \prod_{k=0}^{\infty}\left(1 + x^{2^k} + x^{2\cdot 2^k}\right)$$

① $\left\langle {r \atop n} \right\rangle = [x^n]\, H_r(x)$ (coefficient of x^n in $H_r(x)$)

② $\mathrm{fusc}(n) = [x^{n+1}]\, H(x)$

Hyperbinary partitions

An integer partition of n is hyperbinary if it only has parts of size 2^k for some k, and each part has multiplicity 0, 1, or 2.

Example: $n = 43$

Hyperbinary partitions:

(expressed as weakly decreasing sequences of positive integers)

$(32, 8, 2, 1)$

$(16, 16, 8, 2, 1)$

$(32, 4, 4, 2, 1)$

$(16, 16, 4, 4, 2, 1)$

$(16, 8, 8, 4, 4, 2, 1)$

Hyperbinary partitions

Let $h(n) = \#\{\text{hyperbinary partitions of } n\}$.

Its generating function is

$$H(x) = \sum_{n=0}^{\infty} h(n) x^n = \prod_{k=0}^{\infty} \left(1 + x^{2^k} + x^{2 \cdot 2^k}\right).$$

Theorem [Reznick '90]

For all $n \geq 0$,

$$h(n) = \mathrm{fusc}(n+1).$$

Proof idea: Check recurrence.

(1) If $(\lambda_1, \ldots, \lambda_k)$ is a hyperbinary partition of $2n+1$, then $\lambda_k = 1$ and $\left(\frac{\lambda_1}{2}, \ldots, \frac{\lambda_{k-1}}{2}\right)$ is a hyperbinary partition of n.

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(2) If $(\lambda_1, \dots, \lambda_k)$ is a partition of $2n$, then either $\lambda_k = \lambda_{k-1} = 1$ or $\lambda_k \geq 2$ ...

Example: $n = 43$

$h(43)$	$h(21)$	$h(10)$	$h(5)$	$h(2)$
$(32,8,2,1)$	$(16,4,1)$	$(8,2)$	$(4,1)$	(2)
$(16,16,8,2,1)$	$(8,8,4,1)$	$(4,4,2)$	$(2,2,1)$	$(1,1)$
			$h(4)$	$h(2)$
$(32,4,4,2,1)$	$(16,2,2,1)$	$(8,1,1)$	(4)	(2)
$(16,16,4,4,2,1)$	$(8,8,2,2,1)$	$(4,4,1,1)$	$(2,2)$	$(1,1)$
$(16,8,8,4,4,2,1)$	$(8,4,4,2,2,1)$	$(4,2,2,1,1)$	$(2,1,1)$	$h(1)$
				(1)

Calkin-Wilf sequence

For $n \geq 0$, let $CW(n) = \dfrac{\text{fusc}(n)}{\text{fusc}(n+1)} = \dfrac{h(n-1)}{h(n)}$.

n	$\text{fusc}(n)$	$CW(n)$
0	0	$0/1$
1	1	$1/1$
2	1	$1/2$
3	2	$2/1$
4	1	$1/3$
5	3	$3/2$
6	2	$2/3$
7	3	$3/1$
8	1	$1/4$
9	4	$4/3$
10	3	$3/5$

Calkin-Wilf sequence

For $n \geq 0$, let $CW(n) = \dfrac{\text{fusc}(n)}{\text{fusc}(n+1)} = \dfrac{h(n-1)}{h(n)}$.

Theorem [Calkin-Wilf '00]

For any nonnegative rational number r/s, there is a unique n such that

$$CW(n) = r/s.$$

$CW(0), CW(1), CW(2), CW(3), \ldots$ lists all nonnegative rationals (in reduced form)

Explicitly, we can find $n \in \mathbb{N}$ such that $CW(n) = r/s$ from the continued fraction expansion of r/s.

Continued fractions

For $\frac{r}{s} \geq 0$, there is a unique sequence of integers $a_1, a_2, \ldots, a_{2m-1}$ such that $a_1 \geq 0$, $a_2, \ldots, a_{2m-1} \geq 1$

and
$$\frac{r}{s} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_{2m-1}}}}}.$$

We can assume the sequence has odd length since $a+1 = a + \frac{1}{1}$.

Example

$$\frac{5}{12} = 0 + \cfrac{1}{\frac{12}{5}}$$

$$= 0 + \cfrac{1}{2 + \cfrac{1}{\frac{5}{2}}}$$

$$= 0 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \frac{1}{1}}}}$$

a-sequence
[0, 2, 2, 1, 1]

Convert [0, 2, 2, 1, 1] to binary

$$0 \mid \begin{matrix} 2 \\ 0\ 0 \end{matrix} \mid \begin{matrix} 2 \\ 1\ 1 \end{matrix} \mid \begin{matrix} 1 \\ 0 \end{matrix} \mid \begin{matrix} 1 \\ 1 \end{matrix}$$

reverse

101100 is the binary expansion of $n = 44$

$$CW(44) = \frac{\mathrm{fusc}(44)}{\mathrm{fusc}(45)} = \frac{h(43)}{h(44)} = \frac{5}{12}.$$

q-deformed rationals

For $n \in \mathbb{N}$, set $[n]_q = 1 + q + \cdots + q^{n-1}$.

For rational $\frac{r}{s} \geq 0$ with continued fraction expansion

$$\frac{r}{s} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_{2m}}}}},$$

Morier-Genoud and Ovsienko defined the q-analogue

$$\left[\frac{r}{s}\right]_q = [a_1]_q + \cfrac{q^{a_1}}{[a_2]_{q^{-1}} + \cfrac{q^{-a_2}}{[a_3]_q + \cfrac{q^{a_3}}{\ddots \cfrac{}{[a_{2m-1}]_q + \cfrac{q^{a_{2m-1}}}{[a_{2m}]_{q^{-1}}}}}}}.$$

Example: $$\left[\frac{5}{12}\right]_q = 0 + \cfrac{1}{1 + q^{-1} + \cfrac{q^{-2}}{1 + q + \cfrac{q^2}{1 + q^{-1}}}}$$

$$= \cfrac{1}{1 + q^{-1} + \cfrac{q^{-3} + q^{-2}}{q^{-1} + 2 + q + q^2}}$$

$$= \frac{q^{-1} + 2 + q + q^2}{q^{-3} + 2q^{-2} + 3q^{-1} + 3 + 2q + q^2}$$

Length generating function

Given a partition λ, the length of λ, denoted $\ell(\lambda)$, is the number of parts in λ.

Let $h_q(n) = \sum_{\lambda} q^{\ell(\lambda)}$

where the sum ranges over hyperbinary partitions of n.

Example

$n = 43$

$(32, 8, 2, 1)$ $\quad q^4$

$(16, 16, 8, 2, 1)$ $\quad q^5$

$(32, 4, 4, 2, 1)$ $\quad q^5$

$(16, 16, 4, 4, 2, 1)$ $\quad q^6$

$(16, 8, 8, 4, 4, 2, 1)$ $\quad q^7$

hyperbinary partitions of 43

$$h_q(43) = q^4 + 2q^5 + q^6 + q^7$$

q-Calkin-Wilf

Let $CW_q(n) = \dfrac{h_q(n-1)}{h_q(n)}$.

Example: $CW_q(44) = \dfrac{h_q(43)}{h_q(44)} = \dfrac{q^4+2q^5+q^6+q^7}{q^3+2q^4+3q^5+3q^6+2q^7+q^8}$

Compare to $[CW(44)]_q = \dfrac{q^{-1}+2+q+q^2}{q^{-3}+2q^{-2}+3q^{-1}+3+2q+q^2}$.

q-Calkin-Wilf

Let $CW_q(n) = \dfrac{h_q(n-1)}{h_q(n)}$.

Example: $CW_q(44) = \dfrac{h_q(43)}{h_q(44)} = \dfrac{q^4+2q^5+q^6+q^7}{q^3+2q^4+3q^5+3q^6+2q^7+q^8}$

Compare to $[CW(44)]_q = \dfrac{q^{-1}+2+q+q^2}{q^{-3}+2q^{-2}+3q^{-1}+3+2q+q^2}$.

Theorem A: [MPS '25, Mansour-Shattuck '11]

Generalized Calkin-Wilf tree
- more parameters

For all $n \geq 1$,

$$CW_q(n) = q\,[CW(n)]_q.$$

Proof idea: There is a q-analogue for the recurrence for $h(n)$. Compare with continued fraction recurrence.

(II) A poset of hyperbinary partitions

We partially order partitions of n by refinement.

* $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ refines $\mu = (\mu_1, \ldots, \mu_m)$ if there is an ordered set partition $\mathcal{B} = (B_1, \ldots, B_m)$ of $[\ell]$ such that $\mu_i = \sum_{j \in B_i} \lambda_j$.

Example:

$\lambda = (3, 3, 2, 2, 2, 1, 1)$ refines $\mu = (7, 4, 3)$ since

$\mu_1 = \lambda_1 + \lambda_3 + \lambda_4$, $\quad \mu_2 = \lambda_5 + \lambda_6 + \lambda_7$, $\quad \mu_3 = \lambda_3$.

Ordered set partitions $\qquad\qquad$ Set partitions

Face lattice of permutohedron $\qquad$ lattice of flats

$$\begin{tikzcd}
\Sigma_n \arrow[r] \arrow[d] & \Pi_n \arrow[d] \\
\Lambda_n \arrow[r] & \mathcal{P}_n
\end{tikzcd}$$

Boolean lattice $\qquad\qquad$ not a lattice ?!?

Ordered integer partitions $\qquad\qquad$ Integer partitions

A poset of hyperbinary partitions

Let $\mathcal{H}(n)$ denote the subposet of hyperbinary partitions of n.

Example: $\mathcal{H}(43)$

$$\begin{tikzcd}[column sep=small]
 & (32,8,2,1) = \hat{1}(43) \arrow[dl, dash] \arrow[dr, dash] & \\
(16,16,8,2,1) \arrow[dr, dash] & & (32,4,4,2,1) \arrow[dl, dash] \\
 & (16,16,4,4,2,1) \arrow[d, dash] & \\
 & (16,8,8,4,4,2,1) = \hat{0}(43) &
\end{tikzcd}$$

Theorem B: [MPS '25] (extending [Brunetti, D'Aniello '19])

(i) $\mathcal{H}(n)$ is a distributive lattice.

(ii) There is a unique minimum $\hat{0}(n)$ and maximum $\hat{1}(n)$.

(iii) The rank of λ is $\ell(\hat{0}(n)) - \ell(\lambda)$.

(iv) $\mathcal{H}(n)$ is isomorphic to the lattice of order ideals of a fence poset.

Hyperbinary expansions

The binary expansion of n is

$$\beta(n) = b_1 b_2 \cdots b_k$$

if $b_1 = 1$, $b_i \in \{0,1\}$ for $i \geq 2$,

and $n = b_1 \cdot 2^{k-1} + b_2 \cdot 2^{k-2} + \cdots + b_k$.

A hyperbinary expansion of n is a word

$$d = d_1 d_2 \cdots d_k$$

of the same length as $\beta(n)$ such that

$d_i \in \{0,1,2\}$ for all i, and

$$n = d_1 \cdot 2^{k-1} + d_2 \cdot 2^{k-2} + \cdots + d_k.$$

Let $\mathcal{D}(n)$ be the set of hyperbinary expansions of n.

Then $\mathcal{H}(n) \xrightarrow{\text{bij}} \mathcal{D}(n)$ where

$\lambda \longmapsto d$, d_i = multiplicity of 2^{k-i} in λ

Hyperbinary expansions

The bijection $\mathcal{H}(n) \to \mathcal{D}(n)$ induces a partial order on $\mathcal{D}(n)$.

$\mathcal{H}(43)$

$$
\begin{tikzcd}[column sep=tiny]
 & (32,8,2,1) \arrow[dl, no head] \arrow[dr, no head] & \\
(16,16,8,2,1) \arrow[dr, no head] & & (32,4,4,2,1) \arrow[dl, no head] \\
 & (16,16,4,4,2,1) \arrow[d, no head] & \\
 & (16,8,8,4,4,2,1) &
\end{tikzcd}
$$

$\mathcal{D}(43)$

$$
\begin{tikzcd}[column sep=tiny]
 & 101011 \arrow[dl, no head] \arrow[dr, no head] & \\
021011 \arrow[dr, no head] & & 100211 \arrow[dl, no head] \\
 & 020211 \arrow[d, no head] & \\
 & 012211 &
\end{tikzcd}
$$
.

Lemma: [MPS '25]

The covers in $\mathcal{D}(n)$ are of the form

$$
\begin{tikzcd}
u10v \arrow[d, no head] \\
u02v
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
u20v \arrow[d, no head] \\
u12v
\end{tikzcd}
.
$$

Extrema

The principal prefix of $\beta(n)$ is $b_1 \cdots b_r$ if b_{r+1} is the rightmost 0 in $\beta(n)$.

Example

The principal prefix of $\beta(43) = 101\underline{0}11$ is 101.

Proposition [MPS '25, BD '19]

① The maximum element of $\mathcal{D}(n)$ is the binary expansion.

$$\hat{1}(n) = \beta(n) = b_1 b_2 \cdots b_k$$

② The minimum element of $\mathcal{D}(n)$ is the unique expansion whose zeroes form a prefix.

$$\hat{0}(n) = \begin{cases} 1^k & \text{if } n = 2^k - 1 \\ 0\,(b_2+1)\cdots(b_r+1)\,2\,1^{k-r-1} & \text{if } n \neq 2^k - 1 \end{cases}$$

Fence posets

A **fence** (or **zigzag poset**) is a partial order on a set $\{x_1, \dots, x_r\}$ whose covers are of the form $x_i \lessdot x_{i+1}$ or $x_i \gtrdot x_{i+1}$ for all i.

If $b_1 b_2 \cdots b_r$ is the principal prefix of $\beta(n)$, we define $\mathcal{F}(n)$ to be the fence on $\{x_1, \dots, x_r\}$ such that for $2 \le i \le r$,

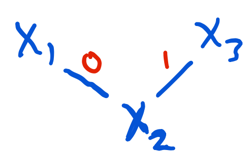
- if $b_i = 0$, then $x_{i-1} \gtrdot x_i$
- if $b_i = 1$, then $x_{i-1} \lessdot x_i$.

Example: $n = 43$

$\beta(43) = 101011$

$r = 3$

$\mathcal{F}(43)$



Order ideals

An order ideal of $\mathcal{F}(n)$ is a downward closed subposet: $y \in I$, $x \leq y \Rightarrow x \in I$.

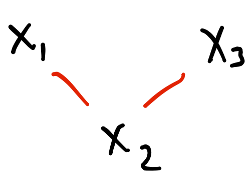
The poset of order ideals $\mathcal{J}(\mathcal{F}(n))$ ordered by inclusion is a distributive lattice. [FTFDL]

$$(I \vee J) \wedge K = (I \wedge K) \vee (J \wedge K)$$

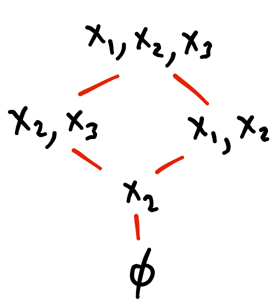
$$(I \wedge J) \vee K = (I \vee K) \wedge (J \vee K)$$

Example

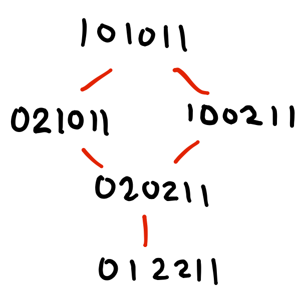
$\mathcal{F}(43)$



$\mathcal{J}(\mathcal{F}(43))$



$\mathcal{D}(43)$



Isomorphism

Theorem B: [MPS '25]

(i) $\mathcal{H}(n)$ is a distributive lattice.

(ii) There is a unique minimum $\hat{0}(n)$ and maximum $\hat{1}(n)$.

(iii) The rank of λ is $\ell(\hat{0}(n)) - \ell(\lambda)$.

(iv) $\mathcal{H}(n)$ is isomorphic to the lattice of order ideals of a fence poset.

Corollary:

If $I \lessdot I \cup \{x_i\}$ is a cover in $\mathcal{J}(\mathcal{F}(n))$,

then the corresponding cover in $\mathcal{D}(n)$ is of the form

$$d_1 \cdots d_{i-1}\,\underline{02}\,d_{i+2} \cdots d_k \lessdot d_1 \cdots d_{i-1}\,\underline{10}\,d_{i+2} \cdots d_k$$

or

$$d_1 \cdots d_{i-1}\,\underline{12}\,d_{i+2} \cdots d_k \lessdot d_1 \cdots d_{i-1}\,\underline{20}\,d_{i+2} \cdots d_k$$

Rank generating function

The rank generating function of $\mathcal{J}(\mathcal{F}(n))$ is

$$\mathrm{rgf}_n(t) = \sum_{I} t^{|I|}.$$

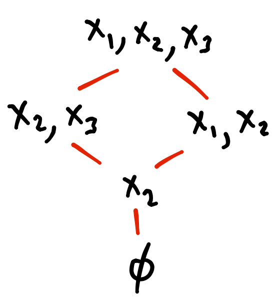
Theorem: [MPS '25]

For all $n \geq 1$, if the principal prefix of $\beta(n)$ has length r and $\beta(n)$ has s ones, then

$$h_q(n) = q^{r+s} \cdot \mathrm{rgf}_n(q^{-1}).$$

Example:

$\mathcal{J}(\mathcal{F}(43))$



q^3

$2q^2$

q

1

$r=3,\ s=4$

$$\mathrm{rgf}_{43}(q) = 1 + q + 2q^2 + q^3$$

$$q^{r+s} \cdot \mathrm{rgf}_{43}(q^{-1}) = q^4 + 2q^5 + q^6 + q^7$$

q-deformed rationals from fences

Corollary: [MPS '25]

For all $n \geq 1$, there exists $a \in \mathbb{Z}$ such that

$$[CW(n)]_q = q^a \frac{rgf_{n-1}(q^{-1})}{rgf_n(q)}.$$

Remark: This result recovers another method of constructing q-deformed rational numbers given in [Morier-Genoud – Ovsienko '20]. Their construction is expressed in terms of closure sets of linear type A quivers, which are essentially equivalent to order ideals of fence posets.

(III) Matrices

Let $L = \begin{bmatrix} 1 & 0 \\ 1 & q^{-1} \end{bmatrix}$ and $R = \begin{bmatrix} q & 1 \\ 0 & 1 \end{bmatrix}$.

We define a matrix $M(n)$ as a product of L's and R's.

Let $\beta(n) = b_1 b_2 \cdots b_k$ be the binary expansion of n.

Reverse the sequence and remove b_1 to get

$$b_k b_{k-1} \cdots b_2 .$$

Replace 0 with L and 1 with R in this sequence, and let $M(n)$ be the product of these matrices.

Example:

$\beta(43) = 101011$

reverse: $1101\,0\cancel{1}$

$$M(43) = RRLRL$$

$$= \begin{bmatrix} q & 1 \\ 0 & 1 \end{bmatrix}^2 \begin{bmatrix} 1 & 0 \\ 1 & q^{-1} \end{bmatrix} \begin{bmatrix} q & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & q^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} q^{-1}+2+2q+2q^2+q^3 & q^{-2}+2q^{-1}+1+q \\ q^{-1}+1+q & q^{-1}+1 \end{bmatrix}$$

Matrices

Theorem C: [MPS '25]

Let $2^k \leq n < 2^{k+1} - 1$, $\beta(n) = b_1 b_2 \cdots b_{k+1}$.

Let j be the largest index such that

$b_1 = b_2 = \cdots = b_j = 1$,

and let $n' \geq 0$ such that $\beta(n') = 1 b_{j+2} b_{j+3} \cdots b_{k+1}$.

Then
$$M(n) = \begin{bmatrix} q^{-k+2j-1} h_q(n'-1) & q^{-k+1} h_q(n-2^k-1) \\ q^{-k+2j-2} h_q(n') & q^{-k} h_q(n-2^k) \end{bmatrix}.$$

If $n = 2^{k+1} - 1$, then

$$M(n) = \begin{bmatrix} q^k & q^{-k+1} h_q(n-2^k-1) \\ 0 & q^{-k} h_q(n-2^k) \end{bmatrix}.$$

Row sums

Theorem: [MPS '25]

If $2^k \le n < 2^{k+1}$, then

$$M(n) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} q^{-k} \, h_q(n-1) \\ q^{-k-1} \, h_q(n) \end{bmatrix}.$$

Corollary:

For all $n \ge 1$,

$[CW(n)]_q$ is the ratio of the two entries

of $M(n) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(IV) Future questions

① Other statistics on hyperbinary partitions

$$\ell(\lambda) = p_1 + 2p_2 \quad \text{where}$$

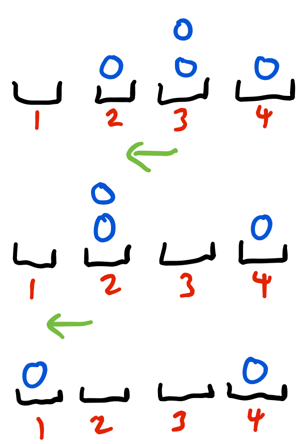
$p_1 = \#$ parts of λ of multiplicity 1

and

$p_2 = \#$ parts of λ of multiplicity 2.

See [Klavžar - Milutinović - Petr '07,
Bates - Mansour '11,
Mansour - Shattuck '11, '15]

② Chip firing ~ other rules/statistics?



Future questions

③ Other subposets of integer partitions?

Ordered set partitions | Set partitions

Face lattice of permutohedron

$$\begin{tikzcd} \Sigma_n \arrow[r] \arrow[d] & \Pi_n \arrow[d] \\ \Lambda_n \arrow[r] & \mathcal{P}_n \end{tikzcd}$$

lattice of flats

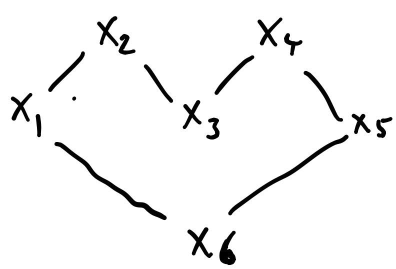
Boolean lattice

not a lattice ?!?

Ordered integer partitions | Integer partitions

④ Circular fences and hyperbinary expansions?

[Oğuz - Ravichandran '23]



THANKS!