A poset of generating functions of partitions of $n$

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Many thanks to Professor Keith for organizing this seminar and for giving me the opportunity to speak!

The following work was done as part of my PhD research under Fabrizio Zanello.

The topic of this talk is a poset, $P_n$, of generating functions of partitions of $n$.

Outline of the talk

- Review of some partition definitions/terminology
- Introduction to $P_n$
- Our main results: two “balancing” theorems
- Other results on $P_n$
- Questions for future work
Definition of a partition

- A *partition* of a positive integer $n$ is a non-increasing sequence of positive integers that sums to $n$.

- **Example.**
  
  $(4, 2, 1, 1)$ is a partition of 8 since $4 + 2 + 1 + 1 = 8$.

- **Example.** All partitions of $n = 5$:
  - $(5)$
  - $(4, 1)$
  - $(3, 2)$
  - $(3, 1, 1)$
  - $(2, 2, 1)$
  - $(2, 1, 1, 1)$
  - $(1, 1, 1, 1, 1)$
Some basic terminology

- If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\kappa)$ is a partition of $n$, then:
  - $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\kappa \geq 1$
  - $\lambda_1 + \lambda_2 + \cdots + \lambda_\kappa = n$
  - The $\lambda_i$ are the *parts* of the partition
  - The *length* of $\lambda$ is $\kappa$ (the number of parts)

- **Example.** The partition $(7, 4, 4, 2, 1)$ has length 5 with part sizes 1, 2, 4 and 7.
The Ferrers diagram of $\lambda = (\lambda_1, \ldots, \lambda_\kappa)$ is a left-justified array of boxes having $\lambda_i$ boxes in the $i^{th}$ row.

The Ferrers diagram of $(7, 4, 4, 2, 1)$ is:

```
<table>
<thead>
<tr>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>1</td>
</tr>
</tbody>
</table>
```
Conjugates

- Reflecting a Ferrers diagram along the main diagonal produces the *conjugate* partition.

- **Example.** The conjugate of \((7, 4, 4, 2, 1)\) is \((5, 4, 3, 3, 1, 1, 1)\)

- The conjugate of \(\lambda\) is denoted by \(\lambda'\).
A partition $\mu$ *fits inside* $\lambda$ if the Ferrers diagram for $\mu$ is entirely contained in that of $\lambda$.

**Example.** $(3, 3, 1)$ and $(5, 4, 4, 1, 1)$ are two (of many) partitions that fit inside $(7, 4, 4, 2, 1)$.
Generating function for a partition $\lambda$

- Given a partition $\lambda$ of $n$, let $a_i$ be the number of partitions of $i$ that fit inside $\lambda$. Then the **generating function** for $\lambda$ is

  \[ G_{\lambda} = 1 + a_1q + a_2q^2 + \cdots + a_nq^n. \]

- The study of these $G_{\lambda}$ was introduced by Stanton in 1990 (*Unimodality and Young’s Lattice*).

- More work in this area has been done by Alpoge, Stanley and Zanello (in the distinct parts case), and Zbarsky.
Given a partition $\lambda$ of $n$, let $a_i$ be the number of partitions of $i$ that fit inside $\lambda$. Then the generating function for $\lambda$ is

$$G_\lambda = 1 + a_1q + a_2q^2 + \cdots + a_nq^n.$$ 

**Example.** Consider $\lambda = (4, 2, 1)$

$$G_{(4, 2, 1)} = 1 + 1q$$
Generating function for a partition $\lambda$

- Given a partition $\lambda$ of $n$, let $a_i$ be the number of partitions of $i$ that fit inside $\lambda$. Then the generating function for $\lambda$ is

$$G_\lambda = 1 + a_1 q + a_2 q^2 + \cdots + a_n q^n.$$ 

- **Example.** Consider $\lambda = (4, 2, 1)$

$$G_{(4, 2, 1)} = 1 + q + 2q^2$$
Generating function for a partition $\lambda$

- Given a partition $\lambda$ of $n$, let $a_i$ be the number of partitions of $i$ that fit inside $\lambda$. Then the generating function for $\lambda$ is

$$G_\lambda = 1 + a_1q + a_2q^2 + \cdots + a_nq^n.$$ 

- **Example.** Consider $\lambda = (4, 2, 1)$

$$G_{(4, 2, 1)} = 1 + q + 2q^2 + 3q^3$$
Generating function for a partition $\lambda$

- Given a partition $\lambda$ of $n$, let $a_i$ be the number of partitions of $i$ that fit inside $\lambda$. Then the *generating function* for $\lambda$ is

$$G_{\lambda} = 1 + a_1 q + a_2 q^2 + \cdots + a_n q^n.$$  

- **Example.** Consider $\lambda = (4, 2, 1)$

$$G_{(4, 2, 1)} = 1 + q + 2q^2 + 3q^3 + 4q^4$$
Generating function for a partition $\lambda$

Given a partition $\lambda$ of $n$, let $a_i$ be the number of partitions of $i$ that fit inside $\lambda$. Then the **generating function** for $\lambda$ is

$$G_\lambda = 1 + a_1q + a_2q^2 + \cdots + a_nq^n.$$  

**Example.** Consider $\lambda = (4, 2, 1)$

$$G_{(4,2,1)} = 1 + q + 2q^2 + 3q^3 + 4q^4 + 4q^5$$

![Diagram of partitions](image-url)
Generating function for a partition $\lambda$

- Given a partition $\lambda$ of $n$, let $a_i$ be the number of partitions of $i$ that fit inside $\lambda$. Then the *generating function* for $\lambda$ is

$$G_\lambda = 1 + a_1 q + a_2 q^2 + \cdots + a_n q^n.$$

- **Example.** Consider $\lambda = (4, 2, 1)$

$$G_{(4, 2, 1)} = 1 + q + 2q^2 + 3q^3 + 4q^4 + 4q^5 + 3q^6$$
Given a partition $\lambda$ of $n$, let $a_i$ be the number of partitions of $i$ that fit inside $\lambda$. Then the generating function for $\lambda$ is

$$G_\lambda = 1 + a_1q + a_2q^2 + \cdots + a_nq^n.$$ 

**Example.** Consider $\lambda = (4, 2, 1)$

$$G_{(4, 2, 1)} = 1 + q + 2q^2 + 3q^3 + 4q^4 + 4q^5 + 3q^6 + q^7$$
A special case: generating functions for rectangles

- Consider a partition of the form \((a, a, \ldots, a)\) (a rectangle). The generating functions for rectangles are the \(q\)-binomial coefficients

\[
\begin{bmatrix} a + b \\ b \end{bmatrix}_q.
\]

- **Example.** The generating function for \((4, 4, 4)\) is \(\left[\begin{array}{c} 4+3 \\ 3 \end{array}\right]_q\).

\[(4, 4, 4): \begin{array}{|c|c|c|}
\hline
1 & 1 & 1 \\
\hline
q & q & q \\
\hline
q^2 & q^2 & q^2 \\
\hline
q^3 & q^3 & q^3 \\
\hline
q^4 & q^4 & q^4 \\
\hline
q^5 & q^5 & q^5 \\
\hline
q^6 & q^6 & q^6 \\
\hline
q^7 & q^7 & q^7 \\
\hline
q^8 & q^8 & q^8 \\
\hline
q^9 & q^9 & q^9 \\
\hline
q^{10} & q^{10} & q^{10} \\
\hline
q^{11} & q^{11} & q^{11} \\
\hline
q^{12} & q^{12} & q^{12} \\
\hline
\end{array}\]

\[G_{(4,4,4)} = \left[\begin{array}{c} 4+3 \\ 3 \end{array}\right]_q = 1 + q + 2q^2 + 3q^3 + 4q^4 + 4q^5 + 5q^6 + 4q^7 + 4q^8 + 3q^9 + 2q^{10} + q^{11} + q^{12} \]
The poset $P_n$

- For a positive integer $n$, define the poset:
  
  $$P_n = \{G_\lambda \mid \lambda \text{ is a partition of } n\}.$$  

- The ordering in $P_n$ is: $G_\eta \leq G_\lambda$ if $G_\lambda - G_\eta$ has nonnegative coefficients.

- **Example.** In $P_{10}$, $G_{(5, 4, 1)} \leq G_{(4, 3, 2, 1)}$ since
  
  $$G_{(4, 3, 2, 1)} - G_{(5, 4, 1)} = q^4 + 2q^6 + 2q^7 + 2q^8 + q^9,$$

  but $G_{(5, 4, 1)}$ and $G_{(4, 2, 2, 2)}$ are incomparable since
  
  $$G_{(4, 2, 2, 2)} - G_{(5, 4, 1)} = q^4 + q^6 - q^9.$$
Motivation behind $P_n$: Bergeron’s Conjecture

- **Bergeron’s Conjecture.** Let $a \leq b \leq c \leq d$ be positive integers such that $ad = bc$. Then

  $$\left[ \begin{array}{c} a + d \\ a \end{array} \right]_q \leq \left[ \begin{array}{c} b + c \\ b \end{array} \right]_q.$$  

- Intuitively, Bergeron’s conjecture posits that more “square-like” rectangles contain more partitions fitting inside (of each size) than “long thin” ones.

- Stanley and Zanello conjectured more: that $\left[ \begin{array}{c} b + c \\ b \end{array} \right]_q - \left[ \begin{array}{c} a + d \\ a \end{array} \right]_q$ is a polynomial with nonnegative, **unimodal** coefficients.

- The poset $P_n$ places **Bergeron’s Conjecture** in a natural framework. Conjecturally, $q$-binomials in $P_n$ form a chain where the associated partitions increase in “squareness”.
Motivation behind $P_n$: Bergeron’s Conjecture

**Example.**

\[
\begin{bmatrix} 8 + 2 \\ 2 \end{bmatrix}_q \leq \begin{bmatrix} 4 + 4 \\ 4 \end{bmatrix}_q \quad \text{(equivalently, } G_{(8, 8)} \leq G_{(4, 4, 4, 4)})
\]

\[
G_{(8, 8)} = \begin{bmatrix} 8+2 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + 2q^3 + 3q^4 + 3q^5 + 4q^6 + 4q^7 + 5q^8 + 4q^9 + 4q^{10} + 3q^{11} + 3q^{12} + 2q^{13} + 2q^{14} + q^{15} + q^{16}
\]

\[
G_{(4, 4, 4, 4)} = \begin{bmatrix} 4+4 \\ 4 \end{bmatrix}_q = 1 + q + 2q^2 + 3q^3 + 5q^4 + 5q^5 + 7q^6 + 7q^7 + 8q^8 + 7q^9 + 7q^{10} + 5q^{11} + 5q^{12} + 3q^{13} + 2q^{14} + q^{15} + q^{16}
\]

\[
\begin{bmatrix} 4+4 \\ 4 \end{bmatrix}_q - \begin{bmatrix} 8+2 \\ 2 \end{bmatrix}_q = q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 3q^8 + 3q^9 + 3q^{10} + 2q^{11} + 2q^{12} + q^{13}
\]
Example: $P_{16}$
Background: hooks

- The $i^{th}$ (principal) hook of a partition is the $(i, i)$ cell together with all cells directly to the right and below.

- Hooks have *arms* and *legs*:
The arms and legs of the principal hooks give the *Frobenius notation*.

**Example.** \( \lambda = (7, 4, 4, 2, 1) \)

The Frobenius notation for \((7, 4, 4, 2, 1)\) is:

\[
\begin{pmatrix}
6 & 2 & 1 \\
4 & 2 & 0
\end{pmatrix} \quad \leftarrow \text{arms}
\]

\[
\begin{pmatrix}
6 & 2 & 1 \\
4 & 2 & 0
\end{pmatrix} \quad \leftarrow \text{legs}
\]
Background: balancing a partition

Given a partition $\lambda = \left( \begin{array}{c} A_k \ A_{k-1} \ \cdots \ A_1 \\ B_k \ B_{k-1} \ \cdots \ B_1 \end{array} \right)$ we define

$\tilde{\lambda} = \left( \begin{array}{c} \tilde{A}_k \ \tilde{A}_{k-1} \ \cdots \ \tilde{A}_1 \\ \tilde{B}_k \ \tilde{B}_{k-1} \ \cdots \ \tilde{B}_1 \end{array} \right)$, where $\tilde{A}_i + \tilde{B}_i = A_i + B_i$, and

$$
\tilde{A}_i = \begin{cases} 
A_i - 1 & \text{if } A_i > B_i + 1 \\
A_i + 1 & \text{if } A_i + 1 < B_i \\
A_i & \text{otherwise.}
\end{cases}
$$

Visually, $\tilde{\lambda}$ is formed by \textit{balancing} the principal hooks of $\lambda$. 
Visually, $\tilde{\lambda}$ is formed by \textit{balancing} the principal hooks of $\lambda$.

\textbf{Example.}

$$
\lambda = \begin{pmatrix}
11 & 4 & 1 \\
7 & 5 & 3 \\
\end{pmatrix} \quad \begin{pmatrix}
10 & 4 & 2 \\
8 & 5 & 2 \\
\end{pmatrix}
$$

When mapping $\lambda \rightarrow \tilde{\lambda}$, we say we are \textit{balancing} $\lambda$. 
The First Balancing Theorem

- $G_\lambda \leq G_{\tilde{\lambda}}$

- This theorem claims that a more “balanced” partition $\tilde{\lambda}$ contains at least as many partitions (of each size) as the original partition $\lambda$.

- We give a sketch of the proof. The full proof is quite long and consists of several supporting lemmas.

- The final key lemma required extensive use of Mathematica to provide a symbolic comparison of certain inequalities (involving multiple parameters that produced thousands of cases to check).
Fix a partition $\lambda = \left( \begin{array}{c} A_k \\ B_k \end{array} \right) \left( \begin{array}{c} A_{k-1} \\ B_{k-1} \end{array} \right) \cdots \left( \begin{array}{c} A_1 \\ B_1 \end{array} \right)$ and consider a partition $\mu = \left( \begin{array}{c} a_\ell \\ b_\ell \end{array} \right) \left( \begin{array}{c} a_{\ell-1} \\ b_{\ell-1} \end{array} \right) \cdots \left( \begin{array}{c} a_1 \\ b_1 \end{array} \right)$ fitting inside $\lambda$. Define the weight vector of $\mu$ by $wt(\mu) = (a_\ell + b_\ell, \ldots, a_1 + b_1)$.

**Example.** Let $\lambda = \left( \begin{array}{cccc} 10 & 8 & 6 & 4 \\ 9 & 6 & 5 & 2 \end{array} \right)$ with $\mu = \left( \begin{array}{cccc} 10 & 5 & 4 & 0 \\ 6 & 5 & 2 & 1 \end{array} \right)$ fitting inside $\lambda$. Then $wt(\mu) = (16, 10, 6, 1)$.

**Lemma 1.** For any partition $\lambda = \left( \begin{array}{c} A_k \\ B_k \end{array} \right) \left( \begin{array}{c} A_{k-1} \\ B_{k-1} \end{array} \right) \cdots \left( \begin{array}{c} A_1 \\ B_1 \end{array} \right)$ and weight vector $w = (w_1, w_2, \ldots, w_k)$, we have

$$\# \{ \mu \leq \lambda \mid wt(\mu) = w \} \leq \# \{ \tilde{\mu} \leq \tilde{\lambda} \mid wt(\tilde{\mu}) = w \}.$$

**Lemma 1** immediately implies the **First Balancing Theorem**: Expand $G_\lambda$ and $G_{\tilde{\lambda}}$ so that each term counts partitions having the same weight vector in $\lambda$ and $\tilde{\lambda}$. 
Sketch of the proof

- To work toward a proof of **Lemma 1**, we introduce a representation of partitions as (strictly NE) paths in $\mathbb{P} \times \mathbb{P}$.

**Example.** Let $\lambda = (\frac{10}{9}, \frac{8}{6}, \frac{6}{5}, \frac{4}{2})$ and take $\mu = (\frac{10}{6}, \frac{5}{5}, \frac{4}{2}, 0)$ with weight vector $\text{wt}(\mu) = (16, 10, 6, 1)$.

The partition $\mu$ can be represented by the path $\begin{pmatrix} 0, 1 \end{pmatrix} - \begin{pmatrix} 4, 2 \end{pmatrix} - \begin{pmatrix} 5, 5 \end{pmatrix} - \begin{pmatrix} 10, 6 \end{pmatrix}$ in $\mathbb{P} \times \mathbb{P}$.
Example. Consider all partitions fitting inside $\lambda = (\frac{10}{9}, \frac{8}{6}, \frac{6}{5}, \frac{4}{2})$ having the same weight vector, $w = (16, 10, 6, 1)$. These partitions can be represented by paths in the following structure:

We call this the trellis for $\lambda$ and $w$, denoted by $T_{\lambda, w}$. 
Example. Consider all partitions fitting inside \( \lambda = (10 \ 8 \ 6 \ 4 \ 9 \ 6 \ 5 \ 2) \) having the same weight vector, \( \mathbf{w} = (16, 10, 6, 1) \). These partitions can be represented by paths in the following structure:

We call this the trellis for \( \lambda \) and \( \mathbf{w} \), denoted by \( T_{\lambda, \mathbf{w}} \).
Example. Consider all partitions fitting inside
\( \lambda = (\frac{10}{9}, \frac{8}{6}, \frac{6}{5}, \frac{4}{2}) \) having the same weight vector,
\( \mathbf{w} = (16, 10, 6, 1) \). These partitions can be represented by
paths in the following structure:

We call this the trellis for \( \lambda \) and \( \mathbf{w} \), denoted by \( T_{\lambda, \mathbf{w}} \).
Sketch of the proof

- In general, for a partition \( \lambda = \left( \begin{array}{c} A_k \\ B_k \\ \vdots \\ A_1 \\ B_1 \end{array} \right) \) and weight vector \( \mathbf{w} = (w_k, \ldots, w_1) \) the trellis \( \mathcal{T}_{\lambda, \mathbf{w}} \) illustrates all possible partitions that fit inside \( \lambda \) and have weight of \( \mathbf{w} \).

- The number of partitions fitting inside \( \lambda \) having weight \( \mathbf{w} \) is equal to the number of valid paths in \( \mathcal{T}_{\lambda, \mathbf{w}} \).

- Thus, **Lemma 1** can be established by proving that balancing the partition \( \lambda \) to \( \tilde{\lambda} \) produces a new trellis with at least as many valid paths.
Example. Taking $\lambda = (\frac{14}{12} \frac{8}{10} \frac{7}{5} \frac{4}{2})$ with $w = (19, 14, 8, 4)$ and balancing to $\tilde{\lambda} = (\frac{13}{13} \frac{9}{9} \frac{6}{6} \frac{3}{3})$ produces the following trellises:
Sketch of the proof

- For a trellis $\mathcal{T}_\lambda, \nu$, let $P^J = \{(x_1, y_1), \ldots, (x_m, y_m)\}$ be the points in the $J$th diagonal, where $x_{i+1} = x_i - 1$.
- Define $C(P^J) = x_1 - y_m$. This function measures the "centeredness" of $P^J$ along the line $y = x$.
- $\tilde{P}^J$ and $C(\tilde{P}^J)$ are defined similarly in $\mathcal{T}_\lambda, \nu$.
- Example.

\[
\begin{align*}
C(P^1) &= 2, & C(P^2) &= 2, & C(P^3) &= -2, & C(P^4) &= 0
\end{align*}
\]
Sketch of the proof

- **Technical Lemma:** After balancing $\lambda \rightarrow \tilde{\lambda}$, we have:

  $$\#P^J \leq \#\tilde{P}^J,$$

  and

  $$|C(P^J)| \geq |C(\tilde{P}^J)|.$$

- In other words, the diagonals in $T_{\tilde{\lambda}, w}$ have at least as many points, and are more centered (if possible) than the corresponding diagonals in $T_{\lambda, w}$.

- We proved this lemma by induction, and used the symbolic comparison features of Mathematica to complete the argument.
Sketch of the proof

- For the diagonal $\mathbf{P}^J$, define the *path-counting vector* $\mathbf{p}^J$, as the vector whose entries are the number of paths from $\mathbf{P}^1$ to $\mathbf{P}^J$. (The vector $\tilde{\mathbf{p}}^J$ is defined similarly for $\tilde{\mathbf{P}}^J$.)

- **Example.** $\mathbf{p}^4 = (9, 17, 23, 26, 18, 10)$ and $\tilde{\mathbf{p}}^4 = (6, 14, 23, 31, 31, 23, 14, 6)$

![Graphs](image)
Sketch of the proof

- Recall the **First Balancing Theorem**:  
  \[ G_\lambda \leq G_{\tilde{\lambda}}. \]

- This follows from **Lemma 1**:  
  For any partition \( \lambda = \left( \begin{array}{c} A_k & A_{k-1} & \cdots & A_1 \\ B_k & B_{k-1} & \cdots & B_1 \end{array} \right) \) and weight vector \( w = (w_1, w_2, \ldots, w_k) \), we have  
  \[ \#\{ \mu \leq \lambda | \text{wt}(\mu) = w \} \leq \#\{ \tilde{\mu} \leq \tilde{\lambda} | \text{wt}(\tilde{\mu}) = w \}. \]

- In the language of the trellis diagrams, Lemma 1 is equivalent to  
  \[ p^k \cdot 1 \leq \tilde{p}^k \cdot 1. \]
Let $\mathbf{x}$ be a vector of the form \((0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)\) (we call such a vector \textit{admissible}.) Define $D(\mathbf{x}) = a - c$. Define $B(\mathbf{x})$, the \textit{centering} of $\mathbf{x}$, to be the admissible vector with numbers $\tilde{a}, \tilde{b} = b, \tilde{c}$ such that

\[
(\tilde{a}, \tilde{c}) = \begin{cases} 
(a - 1, c + 1) & \text{if } a > c \\
(a + 1, c - 1) & \text{if } a < c \\
(a, c) & \text{if } a = c.
\end{cases}
\]

\textbf{Example.} For $a = 4$, $b = 3$, $c = 1$, $\mathbf{x} = (0, 0, 0, 0, 1, 1, 1, 0)$ and $C(\mathbf{x}) = 3$. Then $B(\mathbf{x}) = (0, 0, 0, 1, 1, 1, 0, 0)$ and $B^2(\mathbf{x}) = (0, 0, 1, 1, 1, 0, 0, 0)$. 
**Lemma 2.** Define $p^J$ and $\tilde{p}^J$ as before, and let $d = C(P^J)$. Let $\mathbf{u}$ be any admissible vector with $a + b + c = \#P^J$. Then $p^J \cdot \mathbf{u} \leq \tilde{p}^J \cdot \tilde{\mathbf{u}}$, where $\tilde{\mathbf{u}}$ is defined by the following cases:

- If $d < -1$, then:
  - If $D(\mathbf{u}) > d + 1$, then $\tilde{\mathbf{u}} = \mathbf{u}$.
  - If $d - 1 \leq D(\mathbf{u}) \leq d + 1$, then $\tilde{\mathbf{u}} = B(\mathbf{u})$.
  - If $D(\mathbf{u}) < d - 1$, then $\tilde{\mathbf{u}} = B^2(\mathbf{u})$.

- If $|d| \leq 1$, then:
  - If $d - 1 \leq D(\mathbf{u}) \leq d + 1$, then $\tilde{\mathbf{u}} = \mathbf{u}$.
  - If $D(\mathbf{u}) < d - 1$ or $D(\mathbf{u}) > d + 1$, then $\tilde{\mathbf{u}} = B(\mathbf{u})$.

- If $d > 1$, then:
  - If $D(\mathbf{u}) < d - 1$, then $\tilde{\mathbf{u}} = \mathbf{u}$.
  - If $d - 1 \leq D(\mathbf{u}) \leq d + 1$, then $\tilde{\mathbf{u}} = B(\mathbf{u})$.
  - If $D(\mathbf{u}) > d + 1$, then $\tilde{\mathbf{u}} = B^2(\mathbf{u})$. 
Setting \( u = 1 \) and \( J = k \) in Lemma 2 yields

\[ p^k \cdot 1 \leq \tilde{p}^k \cdot 1. \]

As we saw, this statement is equivalent to Lemma 1, which in turn implies the **The First Balancing Theorem**: \( G_\lambda \leq G_{\bar{\lambda}} \).
The Second Balancing Theorem

- Given a partition \( \lambda = \left( \frac{A_k}{B_k} \frac{A_{k-1}}{B_{k-1}} \cdots \frac{A_1}{B_1} \right) \) we define \( \bar{\lambda} = \left( \frac{\bar{A_k}}{\bar{B_k}} \frac{\bar{A}_{k-1}}{\bar{B}_{k-1}} \cdots \frac{\bar{A}_1}{\bar{B}_1} \right) \), where \( \bar{A}_i = \max(A_i, B_i) \), and \( \bar{B}_i = \min(A_i, B_i) \).

- Visually, \( \bar{\lambda} \) is formed by offsetting the principal hooks of \( \lambda \) in the same direction.

- Example.

\[
\lambda = \begin{pmatrix} 8 & 2 & 1 \\ 5 & 4 & 2 \end{pmatrix} \quad \rightarrow \quad \bar{\lambda} = \begin{pmatrix} 8 & 4 & 2 \\ 5 & 2 & 1 \end{pmatrix}
\]
The Second Balancing Theorem

- **The Second Balancing Theorem**
  \[ G_\lambda \leq G'_{\overline{\lambda}} \]

- We were able to prove this theorem with an injective argument.
The two balancing theorems give a strong necessary condition on what a partition $\lambda$ that yields a maxima in $P_n$ must look like. It must be fully balanced (i.e., $\lambda = \tilde{\lambda}$), with all hooks “offset” in the same direction (i.e., $\lambda = \overline{\lambda}$).

From this restriction, we used a neat connection to the first Rogers-Ramanujan identity to determine an upper bound for the number of maxima in $P_n$.

On the other hand, we also gave a construction showing that the number of maxima goes to infinity as $n \to \infty$.

Unfortunately, we have no guess as to an asymptotic estimate of the number of maxima in $P_n$. 
More results on the poset $P_n$

- It is not hard to see that $G_\lambda = G_{\lambda'}$. We also found an infinite family of non-conjugate (that is, $\lambda \neq \lambda'$) pairs of partitions in $P_n$ having the same generating function.

- We believe the number of such cases is negligible with respect to all partitions of $n$. Under this assumption, we conjecture that $|P_n| \sim \frac{p(n)}{2}$.

- Conditional to this conjecture, we showed that the number of maxima in $P_n$ is negligible with respect to $|P_n|$. 
Questions for future work

- Is there a **combinatorial** proof of the First Balancing Theorem?
- Is it possible to classify all pairs \( \lambda, \eta \) where \( \lambda' \neq \eta \) and \( G_\lambda = G_\eta \)?
- Can some of the ideas used here be modified to make progress on Bergeron’s positivity conjecture?
Thank you!!