A poset of generating functions of partitions of n

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Introduction

- Many thanks to Professor Keith for organizing this seminar and for giving me the opportunity to speak!
- The following work was done as part of my PhD research under Fabrizio Zanello.
- The topic of this talk is a poset, P_n, of generating functions of partitions of n.
- Outline of the talk
  - Review of some partition definitions/terminology
  - Introduction to P_n
  - Our main results: two "balancing" theorems
  - Other results on P_n
  - Questions for future work

Definition of a partition

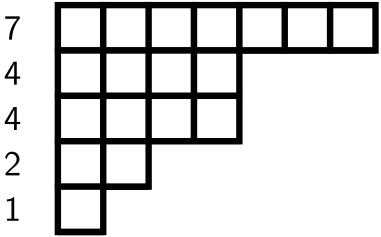
- A *partition* of a positive integer n is a non-increasing sequence of positive integers that sums to n.
- **Example.**
  $(4, 2, 1, 1)$ is a partition of 8 since $4 + 2 + 1 + 1 = 8$.
- **Example.** All partitions of $n = 5$:
  - (5)
  - $(4, 1)$
  - $(3, 2)$
  - $(3, 1, 1)$
  - $(2, 2, 1)$
  - $(2, 1, 1, 1)$
  - $(1, 1, 1, 1, 1)$

Some basic terminology

- If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\kappa)$ is a partition of n, then:
  - $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\kappa \geq 1$
  - $\lambda_1 + \lambda_2 + \cdots + \lambda_\kappa = n$
  - The λ_i are the *parts* of the partition
  - The *length* of λ is κ (the number of parts)
- **Example.** The partition $(7, 4, 4, 2, 1)$ has length 5 with part sizes 1, 2, 4 and 7.

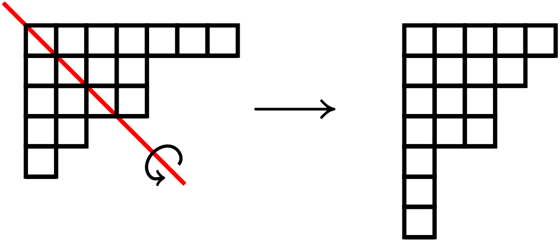
Ferrers diagrams

- The *Ferrers diagram* of $\lambda = (\lambda_1, \ldots \lambda_\kappa)$ is a left-justified array of boxes having λ_i boxes in the i^{th} row.
- The Ferrers diagram of $(7, 4, 4, 2, 1)$ is:



Conjugates

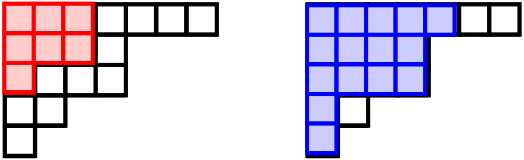
- Reflecting a Ferrers diagram along the main diagonal produces the *conjugate* partition.
- **Example.** The conjugate of $(7, 4, 4, 2, 1)$ is $(5, 4, 3, 3, 1, 1, 1)$



- The conjugate of λ is denoted by λ'.

Partitions inside partitions

- A partition μ *fits inside* λ if the Ferrers diagram for μ is entirely contained in that of λ.
- **Example.** $(3, 3, 1)$ and $(5, 4, 4, 1, 1)$ are two (of many) partitions that fit inside $(7, 4, 4, 2, 1)$



Generating function for a partition λ

- Given a partition λ of n, let a_i be the number of partitions of i that fit inside λ. Then the *generating function* for λ is

$$G_\lambda = 1 + a_1 q + a_2 q^2 + \cdots + a_n q^n.$$

- The study of these G_λ was introduced by Stanton in 1990 (*Unimodality and Young's Lattice*)
- More work in this area has been done by Alpoge, Stanley and Zanello (in the *distinct* parts case), and Zbarsky.

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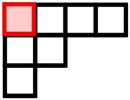
- **Example.** Consider $\lambda = (4, 2, 1)$

$$G_{(4,2,1)} = 1 + \mathbf{1q}$$



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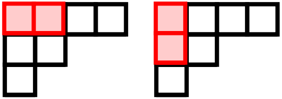
- **Example.** Consider $\lambda = (4, 2, 1)$

$$G_{(4,2,1)} = 1 + q + \mathbf{2q^2}$$



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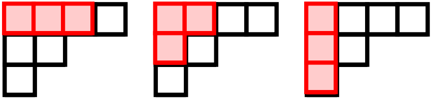
- **Example.** Consider $\lambda = (4, 2, 1)$

$$G_{(4,2,1)} = 1 + q + 2q^2 + \boldsymbol{3q^3}$$



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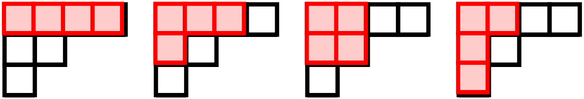

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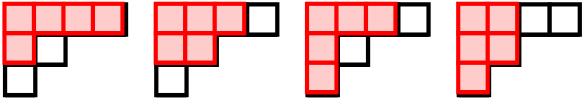

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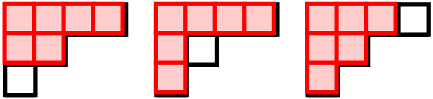
- **Example.** Consider $\lambda = (4, 2, 1)$

$$G_{(4,2,1)} = 1 + q + 2q^2 + 3q^3 + 4q^4 + 4q^5 + \mathbf{3q^6}$$



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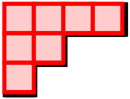
- **Example.** Consider $\lambda = (4, 2, 1)$

$$G_{(4,2,1)} = 1 + q + 2q^2 + 3q^3 + 4q^4 + 4q^5 + 3q^6 + \boldsymbol{q^7}$$

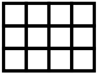


A special case: generating functions for rectangles

- Consider a partition of the form $(\overbrace{a, a, \ldots, a}^{b})$ (a *rectangle*). The generating functions for rectangles are the *q-binomial coefficients*

$$\begin{bmatrix} a+b \\ b \end{bmatrix}_q .$$

- **Example.** The generating function for $(4, 4, 4)$ is $\begin{bmatrix} 4+3 \\ 3 \end{bmatrix}_q$.

$(4, 4, 4)$: 

$$G_{(4,4,4)} = \begin{bmatrix} 4+3 \\ 3 \end{bmatrix}_q = 1 + q + 2q^2 + 3q^3 + 4q^4 + 4q^5 + 5q^6 + 4q^7 + 4q^8 + 3q^9 + 2q^{10} + q^{11} + q^{12}$$

The poset P_n

- For a positive integer n, define the poset:

$$P_n = \{G_\lambda \,|\, \lambda \text{ is a partition of } n\}.$$

- The ordering in P_n is: $G_\eta \le G_\lambda$ if $G_\lambda - G_\eta$ has nonnegative coefficients.
- **Example.** In P_{10}, $G_{(5,4,1)} \le G_{(4,3,2,1)}$ since

$$G_{(4,3,2,1)} - G_{(5,4,1)} = q^4 + 2q^6 + 2q^7 + 2q^8 + q^9,$$

but $G_{(5,4,1)}$ and $G_{(4,2,2,2)}$ are incomparable since

$$G_{(4,2,2,2)} - G_{(5,4,1)} = q^4 + q^6 - q^9.$$

Motivation behind P_n: Bergeron's Conjecture

- **Bergeron's Conjecture.** Let $a \leq b \leq c \leq d$ be positive integers such that $ad = bc$. Then

$$\begin{bmatrix} a+d \\ a \end{bmatrix}_q \leq \begin{bmatrix} b+c \\ b \end{bmatrix}_q .$$

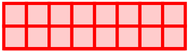
- Intuitively, Bergeron's conjecture posits that more "square-like" rectangles contain more partitions fitting inside (of *each* size) than "long thin" ones.
- Stanley and Zanello conjectured more: that $\begin{bmatrix} b+c \\ b \end{bmatrix}_q - \begin{bmatrix} a+d \\ a \end{bmatrix}_q$ is a polynomial with nonnegative, **unimodal** coefficients.
- The poset P_n places **Bergeron's Conjecture** in a natural framework. Conjecturally, q-binomials in P_n form a chain where the associated partitions increase in "squareness".

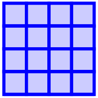
Motivation behind P_n: Bergeron's Conjecture

- **Example.**

$$\begin{bmatrix} 8+2 \\ 2 \end{bmatrix}_q \leq \begin{bmatrix} 4+4 \\ 4 \end{bmatrix}_q$$

(equivalently, $G_{(8,8)} \leq G_{(4,4,4,4)}$)



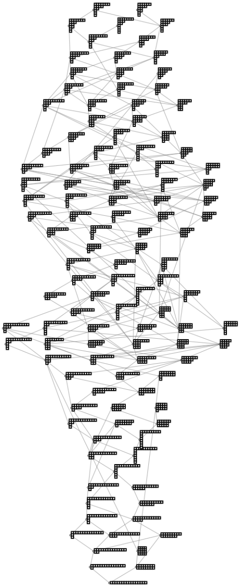


$$G_{(8,8)} = \begin{bmatrix} 8+2 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + 2q^3 + 3q^4 + 3q^5 + 4q^6 + 4q^7 + 5q^8 + 4q^9 + 4q^{10} + 3q^{11} + 3q^{12} + 2q^{13} + 2q^{14} + q^{15} + q^{16}$$

$$G_{(4,4,4,4)} = \begin{bmatrix} 4+4 \\ 4 \end{bmatrix}_q = 1 + q + 2q^2 + 3q^3 + 5q^4 + 5q^5 + 7q^6 + 7q^7 + 8q^8 + 7q^9 + 7q^{10} + 5q^{11} + 5q^{12} + 3q^{13} + 2q^{14} + q^{15} + q^{16}$$

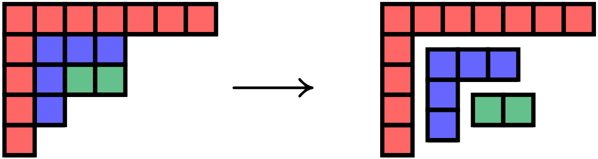
$$\begin{bmatrix} 4+4 \\ 4 \end{bmatrix}_q - \begin{bmatrix} 8+2 \\ 2 \end{bmatrix}_q = q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 3q^8 + 3q^9 + 3q^{10} + 2q^{11} + 2q^{12} + q^{13}$$

Example: P_{16}

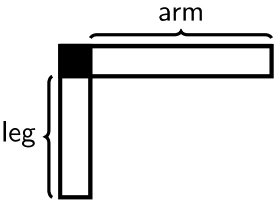


Background: hooks

- The i^{th}(principal) *hook* of a partition is the (i, i) cell together with all cells directly to the right and below.

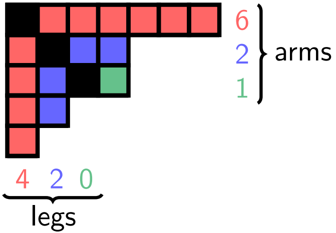


- Hooks have *arms* and *legs*:



Background: Frobenius notation

- The arms and legs of the principal hooks give the *Frobenius notation*.
- **Example.** $\lambda = (7, 4, 4, 2, 1)$



The Frobenius notation for $(7, 4, 4, 2, 1)$ is:

$$\begin{pmatrix} 6 & 2 & 1 \\ 4 & 2 & 0 \end{pmatrix} \begin{matrix} \leftarrow \text{arms} \\ \leftarrow \text{legs} \end{matrix}$$

Background: balancing a partition

- Given a partition $\lambda = \begin{pmatrix} A_k & A_{k-1} & \cdots & A_1 \\ B_k & B_{k-1} & \cdots & B_1 \end{pmatrix}$ we define $\tilde{\lambda} = \begin{pmatrix} \tilde{A}_k & \tilde{A}_{k-1} & \cdots & \tilde{A}_1 \\ \tilde{B}_k & \tilde{B}_{k-1} & \cdots & \tilde{B}_1 \end{pmatrix}$, where $\tilde{A}_i + \tilde{B}_i = A_i + B_i$, and

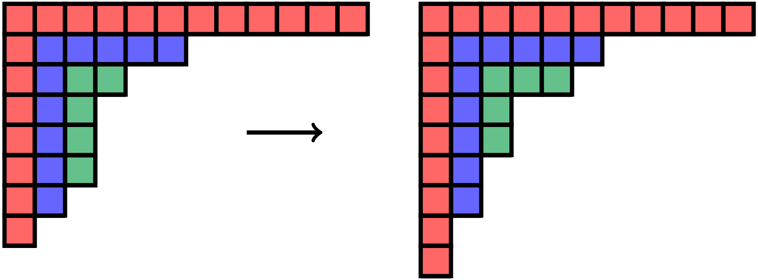
$$\tilde{A}_i = \begin{cases} A_i - 1 & \text{if } A_i > B_i + 1 \\ A_i + 1 & \text{if } A_i + 1 < B_i \\ A_i & \text{otherwise.} \end{cases}$$

- Visually, $\tilde{\lambda}$ is formed by *balancing* the principal hooks of λ.

Background: balancing a partition

- Visually, $\tilde{\lambda}$ is formed by *balancing* the principal hooks of λ.
- **Example.**

$$\lambda = \begin{pmatrix} 11 & 4 & 1 \\ 7 & 5 & 3 \end{pmatrix} \longrightarrow \tilde{\lambda} = \begin{pmatrix} 10 & 4 & 2 \\ 8 & 5 & 2 \end{pmatrix}$$



- When mapping $\lambda \to \tilde{\lambda}$, we say we are *balancing* λ.

The First Balancing Theorem

- **The First Balancing Theorem**

$$G_\lambda \leq G_{\tilde{\lambda}}$$

- This theorem claims that a more "balanced" partition $\tilde{\lambda}$ contains at least as many partitions (*of each size*) as the original partition λ.
- We give a sketch of the proof. The full proof is quite long and consists of several supporting lemmas.
- The final key lemma required extensive use of Mathematica to provide a symbolic comparison of certain inequalities (involving multiple parameters that produced thousands of cases to check).

Sketch of the proof

- Fix a partition $\lambda = \begin{pmatrix} A_k & A_{k-1} & \cdots & A_1 \\ B_k & B_{k-1} & \cdots & B_1 \end{pmatrix}$ and consider a partition $\mu = \begin{pmatrix} a_\ell & a_{\ell-1} & \cdots & a_1 \\ b_\ell & b_{\ell-1} & \cdots & b_1 \end{pmatrix}$ fitting inside λ. Define the *weight vector* of μ by $\mathrm{wt}(\mu) = (a_\ell + b_\ell, \ldots, a_1 + b_1)$.
- **Example.** Let $\lambda = \begin{pmatrix} 10 & 8 & 6 & 4 \\ 9 & 6 & 5 & 2 \end{pmatrix}$ with $\mu = \begin{pmatrix} 10 & 5 & 4 & 0 \\ 6 & 5 & 2 & 1 \end{pmatrix}$ fitting inside λ. Then $\mathrm{wt}(\mu) = (16, 10, 6, 1)$.
- **Lemma 1.** For any partition $\lambda = \begin{pmatrix} A_k & A_{k-1} & \cdots & A_1 \\ B_k & B_{k-1} & \cdots & B_1 \end{pmatrix}$ and weight vector $\mathbf{w} = (w_1, w_2, \ldots, w_k)$, we have

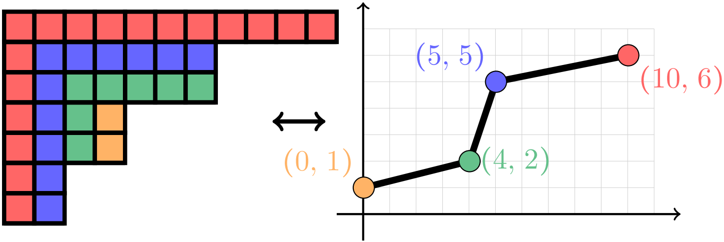
$$\#\{\mu \leq \lambda \,|\, \mathrm{wt}(\mu) = \mathbf{w}\} \leq \#\{\tilde{\mu} \leq \tilde{\lambda} \,|\, \mathrm{wt}(\tilde{\mu}) = \mathbf{w}\}.$$

- **Lemma 1** immediately implies the **First Balancing Theorem**: Expand G_λ and $G_{\tilde{\lambda}}$ so that each term counts partitions having the same weight vector in λ and $\tilde{\lambda}$.

Sketch of the proof

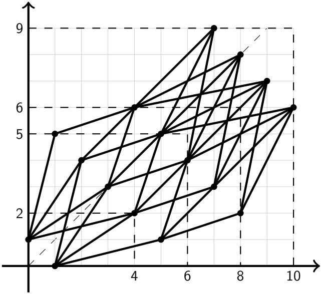
- To work toward a proof of **Lemma 1**, we introduce a representation of partitions as (strictly NE) paths in $\mathbb{P} \times \mathbb{P}$.
- **Example.** Let $\lambda = \left(\begin{smallmatrix} 10 & 8 & 6 & 4 \\ 9 & 6 & 5 & 2 \end{smallmatrix}\right)$ and take $\mu = \left(\begin{smallmatrix} 10 & 5 & 4 & 0 \\ 6 & 5 & 2 & 1 \end{smallmatrix}\right)$ with weight vector $\mathrm{wt}(\mu) = (16, 10, 6, 1)$.

The partition μ can be represented by the path $(0, 1) - (4, 2) - (5, 5) - (10, 6)$ in $\mathbb{P} \times \mathbb{P}$.



Sketch of the proof

- **Example.** Consider **all** partitions fitting inside $\lambda = \left(\begin{smallmatrix} 10 & 8 & 6 & 4 \\ 9 & 6 & 5 & 2 \end{smallmatrix}\right)$ having the same weight vector, $\mathbf{w} = (16, 10, 6, 1)$. These partitions can be represented by paths in the following structure:



- We call this the *trellis* for λ and $\mathbf{w}$, denoted by $\mathcal{T}_{\lambda,\mathbf{w}}$.

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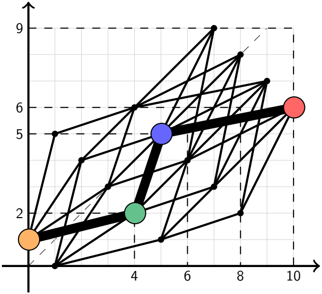

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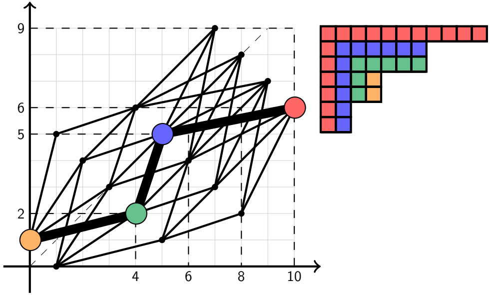


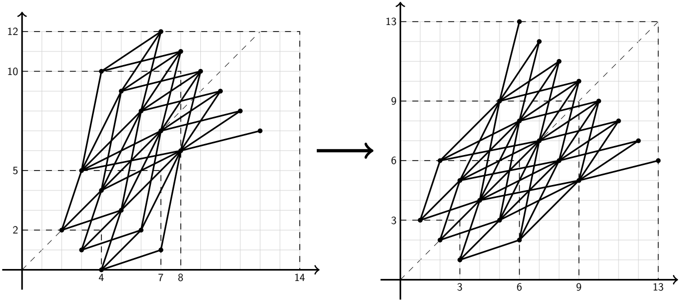
- We call this the *trellis* for λ and $\mathbf{w}$, denoted by $\mathcal{T}_{\lambda,\mathbf{w}}$.

Sketch of the proof

- In general, for a partition $\lambda = \begin{pmatrix} A_k & A_{k-1} & \cdots & A_1 \\ B_k & B_{k-1} & \cdots & B_1 \end{pmatrix}$ and weight vector $\mathbf{w} = (w_k, \ldots, w_1)$ the *trellis* $\mathcal{T}_{\lambda,\mathbf{w}}$ illustrates all possible partitions that fit inside λ and have weight of $\mathbf{w}$.
- The number of partitions fitting inside λ having weight $\mathbf{w}$ is equal to the number of valid paths in $\mathcal{T}_{\lambda,\mathbf{w}}$.
- Thus, **Lemma 1** can be established by proving that balancing the partition λ to $\tilde{\lambda}$ produces a new trellis with at least as many valid paths.

Sketch of the proof

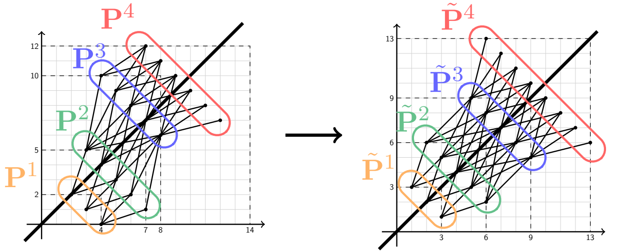
- **Example.** Taking $\lambda = \left(\begin{smallmatrix} 14 & 8 & 7 & 4 \\ 12 & 10 & 5 & 2 \end{smallmatrix}\right)$ with $\mathbf{w} = (19, 14, 8, 4)$ and balancing to $\tilde{\lambda} = \left(\begin{smallmatrix} 13 & 9 & 6 & 3 \\ 13 & 9 & 6 & 3 \end{smallmatrix}\right)$ produces the following trellises:



$\mathcal{T}_{\lambda, \mathbf{w}}$ $\qquad$ $\mathcal{T}_{\tilde{\lambda}, \mathbf{w}}$

Sketch of the proof

- For a trellis $\mathcal{T}_{\lambda,\mathbf{w}}$, let $\mathbf{P}^J = \{(x_1, y_1), \ldots, (x_m, y_m)\}$ be the points in the J^{th} *diagonal*, where $x_{i+1} = x_i - 1$.
- Define $C(\mathbf{P}^J) = x_1 - y_m$. This function measures the "centeredness" of $\mathbf{P}^J$ along the line $y = x$.
- $\tilde{\mathbf{P}}^J$ and $C(\tilde{\mathbf{P}}^J)$ are defined similarly in $\mathcal{T}_{\tilde{\lambda},\mathbf{w}}$.
- **Example.**



$$C(\mathbf{P}^1) = 2, \quad C(\mathbf{P}^2) = 2, \quad C(\mathbf{P}^3) = -2, \quad C(\mathbf{P}^4) = 0$$

Sketch of the proof

- **Technical Lemma:** After balancing $\lambda \to \tilde{\lambda}$, we have:

$$\#\mathbf{P}^J \leq \#\tilde{\mathbf{P}}^J,$$

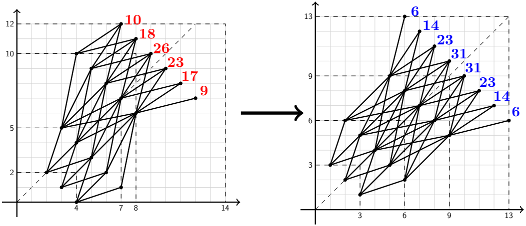
  and

$$|C(\mathbf{P}^J)| \geq |C(\tilde{\mathbf{P}}^J)|.$$

- In other words, the diagonals in $\mathcal{T}_{\tilde{\lambda},\mathbf{w}}$ have at least as many points, and are more centered (if possible) than the corresponding diagonals in $\mathcal{T}_{\lambda,\mathbf{w}}$.
- We proved this lemma by induction, and used the symbolic comparison features of Mathematica to complete the argument.

Sketch of the proof

- For the diagonal $\mathbf{P}^J$, define the *path-counting vector*, $\mathbf{p}^J$, as the vector whose entries are the number of paths from $\mathbf{P}^1$ to $\mathbf{P}^J$. (The vector $\tilde{\mathbf{p}}^J$ is defined similarly for $\tilde{\mathbf{P}}^J$.)
- **Example.** $\mathbf{p}^4 = (9, 17, 23, 26, 18, 10)$ and $\tilde{\mathbf{p}}^4 = (6, 14, 23, 31, 31, 23, 14, 6)$



$\mathcal{T}_{\lambda,\mathbf{w}}$ $\mathcal{T}_{\tilde{\lambda},\mathbf{w}}$

Sketch of the proof

- Recall the **First Balancing Theorem**:

$$G_\lambda \le G_{\tilde{\lambda}}.$$

- This follows from **Lemma 1**:
For any partition $\lambda = \begin{pmatrix} A_k & A_{k-1} & \cdots & A_1 \\ B_k & B_{k-1} & \cdots & B_1 \end{pmatrix}$ and weight vector $\mathbf{w} = (w_1, w_2, \ldots, w_k)$, we have

$$\#\{\mu \le \lambda \,|\, \mathrm{wt}(\mu) = \mathbf{w}\} \le \#\{\tilde{\mu} \le \tilde{\lambda} \,|\, \mathrm{wt}(\tilde{\mu}) = \mathbf{w}\}.$$

- **In the language of the trellis diagrams, Lemma 1 is equivalent to**

$$\mathbf{p}^k \cdot \mathbf{1} \le \tilde{\mathbf{p}}^k \cdot \mathbf{1}.$$

Sketch of the proof

- Let $\mathbf{x}$ be a vector of the form $(\overbrace{0,\ldots,0}^{a\geq 0},\ \overbrace{1,\ldots,1}^{b\geq 1},\ \overbrace{0,\ldots,0}^{c\geq 0})$ (we call such a vector *admissible*.) Define $D(\mathbf{x}) = a - c$. Define $B(\mathbf{x})$, the *centering* of $\mathbf{x}$, to be the admissible vector with numbers $\tilde{a}$, $\tilde{b} = b$, $\tilde{c}$ such that

$$(\tilde{a}, \tilde{c}) = \begin{cases} (a-1,\ c+1) & \text{if } a > c \\ (a+1,\ c-1) & \text{if } a < c \\ (a,\ c) & \text{if } a = c. \end{cases}$$

- **Example.** For $a = 4$, $b = 3$, $c = 1$, $\mathbf{x} = (0,0,0,0,1,1,1,0)$ and $C(\mathbf{x}) = 3$. Then $B(\mathbf{x}) = (0,0,0,1,1,1,0,0)$ and $B^2(\mathbf{x}) = (0,0,1,1,1,0,0,0)$.

Sketch of the proof

- **Lemma 2.** Define $\mathbf{p}^J$ and $\tilde{\mathbf{p}}^J$ as before, and let $d = C(\mathbf{P}^J)$. Let $\mathbf{u}$ be any admissible vector with $a + b + c = \#\mathbf{P}^J$. Then $\mathbf{p}^J \cdot \mathbf{u} \leq \tilde{\mathbf{p}}^J \cdot \tilde{\mathbf{u}}$, where $\tilde{\mathbf{u}}$ is defined by the following cases:
  - If $d < -1$, then:
    - If $D(\mathbf{u}) > d + 1$, then $\tilde{\mathbf{u}} = \mathbf{u}$.
    - If $d - 1 \leq D(\mathbf{u}) \leq d + 1$, then $\tilde{\mathbf{u}} = B(\mathbf{u})$.
    - If $D(\mathbf{u}) < d - 1$, then $\tilde{\mathbf{u}} = B^2(\mathbf{u})$.
  - If $|d| \leq 1$, then:
    - If $d - 1 \leq D(\mathbf{u}) \leq d + 1$, then $\tilde{\mathbf{u}} = \mathbf{u}$.
    - If $D(\mathbf{u}) < d - 1$ or $D(\mathbf{u}) > d + 1$, then $\tilde{\mathbf{u}} = B(\mathbf{u})$.
  - If $d > 1$, then:
    - If $D(\mathbf{u}) < d - 1$, then $\tilde{\mathbf{u}} = \mathbf{u}$.
    - If $d - 1 \leq D(\mathbf{u}) \leq d + 1$, then $\tilde{\mathbf{u}} = B(\mathbf{u})$.
    - If $D(\mathbf{u}) > d + 1$, then $\tilde{\mathbf{u}} = B^2(\mathbf{u})$.

Sketch of the proof

- Setting $\mathbf{u} = \mathbf{1}$ and $J = k$ in **Lemma 2** yields

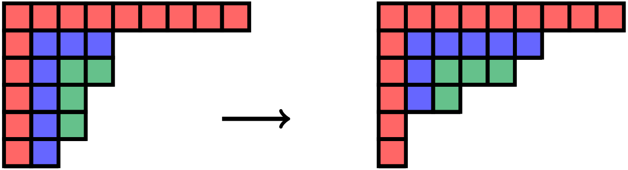
$$\mathbf{p}^k \cdot \mathbf{1} \leq \tilde{\mathbf{p}}^k \cdot \mathbf{1}.$$

- As we saw, this statement is equivalent to **Lemma 1**, which in turn implies the **The First Balancing Theorem**: $G_\lambda \leq G_{\tilde{\lambda}}$.

The Second Balancing Theorem

- Given a partition $\lambda = \begin{pmatrix} A_k & A_{k-1} & \cdots & A_1 \\ B_k & B_{k-1} & \cdots & B_1 \end{pmatrix}$ we define $\overline{\lambda} = \begin{pmatrix} \overline{A}_k & \overline{A}_{k-1} & \cdots & \overline{A}_1 \\ \overline{B}_k & \overline{B}_{k-1} & \cdots & \overline{B}_1 \end{pmatrix}$, where $\overline{A}_i = \max(A_i, B_i)$, and $\overline{B}_i = \min(A_i, B_i)$.
- Visually, $\overline{\lambda}$ is formed by offsetting the principal hooks of λ in the *same* direction.
- **Example.**

$$\lambda = \begin{pmatrix} 8 & 2 & 1 \\ 5 & 4 & 2 \end{pmatrix} \longrightarrow \overline{\lambda} = \begin{pmatrix} 8 & 4 & 2 \\ 5 & 2 & 1 \end{pmatrix}$$



The Second Balancing Theorem

- **The Second Balancing Theorem**

$$G_\lambda \leq G_{\overline{\lambda}}$$

- We were able to prove this theorem with an injective argument.

More results on the poset P_n

- The two balancing theorems give a strong necessary condition on what a partition λ that yields a maxima in P_n must look like. It must be *fully balanced* (i.e., $\lambda = \tilde{\lambda}$), with all hooks "offset" in the *same direction* (i.e., $\lambda = \overline{\lambda}$).
- From this restriction, we used a neat connection to the *first Rogers-Ramanujan identity* to determine an upper bound for the number of maxima in P_n.
- On the other hand, we also gave a construction showing that the number of maxima goes to infinity as $n \to \infty$.
- Unfortunately, we have no guess as to an asymptotic estimate of the number of maxima in P_n.

More results on the poset P_n

- It is not hard to see that $G_\lambda = G_{\lambda'}$. We also found an infinite family of *non-conjugate* (that is, $\lambda \neq \lambda'$) pairs of partitions in P_n having the same generating function.
- We believe the number of such cases is negligible with respect to all partitions of n. Under this assumption, we conjecture that $|P_n| \sim \frac{p(n)}{2}$.
- Conditional to this conjecture, we showed that the number of maxima in P_n is negligible with respect to $|P_n|$.

Questions for future work

- Is there a **combinatorial** proof of the First Balancing Theorem?
- Is it possible to classify all pairs λ, η where $\lambda' \neq \eta$ and $G_\lambda = G_\eta$?
- Can some of the ideas used here be modified to make progress on Bergeron's positivity conjecture?

The End

Thank you!!