Distribution of the sum of reciprocal parts for distinct parts partitions

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This talk is based on:

- W. Bridges, "Distribution of the sum of reciprocal parts for distinct parts partitions," submitted. arXiv:2503.03899

Distinct parts partitions

Definition

A **distinct parts partition** λ of size $|\lambda| = n$ is a sequence of integers satisfying

$$\lambda_1 > \cdots > \lambda_\ell > 0, \qquad \text{and} \qquad \sum_{j=1}^{\ell} \lambda_j = n.$$

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Example

$d(5) = 3$:

$$5, \qquad 4+1, \qquad 3+2.$$

Egyptian fractions

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Given a distinct parts partition λ, the **sum of reciprocal parts** is denoted

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General references on Egyptian fractions:

- P. Erdős and R. L. Graham, Old and New Problems in Combinatorial Number Theory, L'Enseignement Mathématique Université de Genève, 1980.
- R. Guy, Unsolved Problems in Number Theory, 3rd edition, Springer-Verlag, 2004.

Question (Kim–Kim, JCTA (2025))

How is S distributed among distinct parts partitions of n, as $n \to \infty$?

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Recall: $1 + \frac{1}{2} + \cdots + \frac{1}{n} = \log n + \gamma + o(1)$, so clearly

$$0 \le S(\lambda) \le \log n + \gamma + o(1),$$

for all $\lambda \in \mathcal{D}_n$.

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Theorem (Kim–Kim, JCTA (2025))

As $n \to \infty$,

$$\sum_{\lambda \in \mathcal{D}_n} S(\lambda) = d(n) \left(\frac{\log(\sqrt{3n})}{2} + O(n^{-1/2}) \right) \tag{1}$$

$$\sum_{\lambda \in \mathcal{D}_n} S(\lambda)^2 = d(n) \left(\frac{\log^2(\sqrt{3n})}{4} + \frac{\pi^2}{24} + O(n^{-1/2}) \right). \tag{2}$$

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Corollary

$2S - \log(\sqrt{3n})$ *has asymptotic mean 0 and variance* $\frac{\pi^2}{12}$.

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Remark

Bringmann–Kim–Kim (2025+) proved Rademacher-type asymptotic series for (1) and (2) with $O(\sqrt{n})$ error!

Notation: Uniform measure

$P_n :=$ uniform probability measure on distinct parts partitions of n

Example

$d(5) = 3$, so

$$P_n(5) = P_n(4+1) = P_n(3+2) = \frac{1}{3}.$$

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Kim–Kim's Theorem + Chebyshev's inequality: For all $M > 0$,

$$P_n\left(|2S - \log(\sqrt{3n})| > M\right) \leq \frac{\pi^2}{12M^2} \text{ for sufficiently large } n.$$

Random Harmonic Sum

Let $\{\varepsilon_k\}_{k\geq 1}$ be independent random variables with[1]

$$P(\varepsilon_k = \pm 1) = \frac{1}{2}.$$

[1]Throughout, P denotes a probability measure induced by the random variables in its argument.

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Let $\{\varepsilon_k\}_{k\geq 1}$ be independent random variables with[1]

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Definition

The **random harmonic sum** is

$$H := \sum_{k\geq 1} \frac{\varepsilon_k}{k}.$$

Fact

H converges almost surely.

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Main Theorem

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Theorem (B. (2025+))

For any $x \in \mathbb{R}$,

$$\lim_{n\to\infty} P_n(2S - \log(\sqrt{3n}) \leq x) = P(H \leq x).$$

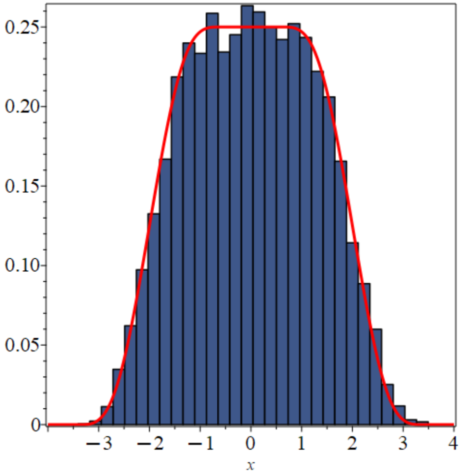

Figure: A histogram of 10 000 values of $2S(\lambda) - \log(\sqrt{3|\lambda|})$, where partitions λ have been generated in Maple by a **Boltzmann sampler** with parameter $q = e^{-\frac{\pi}{\sqrt{12n}}}$ with $n = 2000$. In red is an approximation to the density for H.

Density for the random harmonic sum: Method 1

B. Schmuland, *Random harmonic series*, Amer. Math. Monthly **110** (2003).

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- Each ε_k has characteristic function $\cos(t)$, so $H = \sum \frac{\varepsilon_k}{k}$ has characteristic function
$$\prod_{k\geq 1} \cos\left(\frac{t}{k}\right).$$

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- Each ε_k has characteristic function $\cos(t)$, so $H = \sum \frac{\varepsilon_k}{k}$ has characteristic function
$$\prod_{k\geq 1} \cos\left(\frac{t}{k}\right).$$
- By Fourier inversion, the density is
$$f_H(x) := \frac{1}{\pi}\int_0^\infty \cos(xt)\prod_{k\geq 1}\cos\left(\frac{t}{k}\right)dt.$$
- The product converges very slowly! There is an easier method!

Density for the random harmonic sum: Method 2

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- Write

$$\begin{aligned} H = \quad & \frac{\varepsilon_1}{1} + \frac{\varepsilon_2}{2} + \frac{\varepsilon_4}{4} + \dots \\ & + \frac{\varepsilon_3}{3} + \frac{\varepsilon_6}{6} + \frac{\varepsilon_{12}}{12} + \dots \\ & + \frac{\varepsilon_5}{5} + \frac{\varepsilon_{10}}{10} + \frac{\varepsilon_{20}}{20} + \dots \\ & \vdots \qquad\qquad =: \sum_{j \geq 0} U_j. \end{aligned}$$

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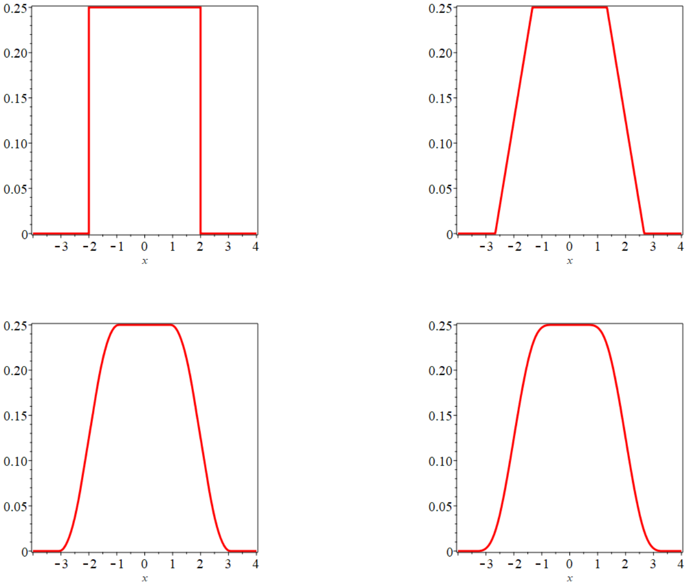
- Each U_j is has uniform distribution on $\left[-\frac{2}{2j+1}, \frac{2}{2j+1}\right]$, so

$$f_H(x) = \left(\frac{1}{4}1_{[-2,2]}\right) * \left(\frac{3}{4}1_{[-\frac{2}{3},\frac{2}{3}]}\right) * \left(\frac{5}{4}1_{[-\frac{2}{5},\frac{2}{5}]}\right) * \cdots$$



Above: Densities for $\sum_{j=0}^{n} U_j$ for $n \in \{0, 1, 2, 3\}$.

Proof Outline

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$$S = \sum_{k=1}^{n} \frac{X_k}{k},$$
where $X_k(\lambda) \in \{0, 1\}$ is the multiplicity of k in λ.

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- Break the sum up into three ranges:
$$[1, n] = \underbrace{[1, k_n]}_{\text{small parts}} \cup \underbrace{(k_n, K_n]}_{\text{intermediate}} \cup \underbrace{(K_n, n]}_{\text{large}},$$
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- Work of Fristedt (1993) gives joint distributions for small and intermediate part sizes.
- We analyze the contribution from large parts by proving a strong version of the **limit shape** for distinct parts partitions.

Proof Outline

Proposition (Small parts)

For any $x \in \mathbb{R}$,

$$\lim_{n\to\infty} P_n\left(\sum_{k\leq k_n} \frac{2X_k}{k} - \log(k_n) - \gamma \leq x\right) = P(H \leq x).$$

Proposition (Intermediate parts)

$$\lim_{n\to\infty} P_n\left(\left|\sum_{k_n<k\leq K_n} \frac{2X_k}{k} - \log\left(\frac{K_n}{k_n}\right)\right| \leq n^{-\frac{1}{11}}\right) = 1.$$

Proposition (Large parts)

$$\lim_{n\to\infty} P_n\left(\left|\sum_{K_n<k\leq n} \frac{2X_k}{k} - \log\left(\frac{\sqrt{3n}}{K_n}\right) + \gamma\right| \leq n^{-\frac{1}{30}}\right) = 1.$$

Small parts: sketch

Small parts behave like independent Bernoulli random variables:

Proposition (Fristedt, (1993) Trans. AMS)

Let $x_k \in \{0, 1\}$ *for* $k = 1, \ldots, k_n$ *with* $k_n = o(n^{1/4})$*, then*

$$\lim_{n\to\infty} \left(P_n \left(X_k = x_k,\ k = 1, \ldots, k_n \right) - \frac{1}{2^{k_n}} \right) = 0.$$

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- Thus,

$$\sum_{k \le k_n} \frac{2X_k}{k} - \log(k_n) - \gamma \approx \sum_{k \le k_n} \frac{2X_k - 1}{k} \approx \sum_{k \le k_n} \frac{\varepsilon_k}{k} \approx H$$

Intermediate parts: sketch

- Using work of Fristedt (1993), one can show

$$\mathrm{E}_n\left(\sum_{k_n<k\le K_n}\frac{2X_k}{k}\right)=\log\left(\frac{K_n}{k_n}\right)+O(n^{-1/6}),$$

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- Chebyshev's Inequality implies that intermediate parts contribute only to the mean of S:

$$P_n\left(\left|\sum_{k_n<k\leq K_n}\frac{2X_k}{k}-\log\left(\frac{K_n}{k_n}\right)\right|>n^{-\frac{1}{11}}\right)\ll\frac{n^{\frac{2}{11}}}{n^{\frac{1}{5}}}=o(1).$$

Large parts: limit shape

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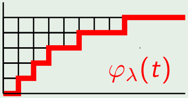
- The **shape** of the Young/Ferrer's diagram for λ is described by the step function,

$$\varphi_\lambda(t) := \sum_{k \le t} X_k(\lambda).$$

Example

Let $\lambda = 8 + 5 + 3 + 2 + 1$.

t	0	1	2	3	4	5	6	7	8	9	...
$\varphi_\lambda(t)$	0	1	2	3	3	4	4	4	5	5	...



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$$\varphi_\lambda(t) := \sum_{k \le t} X_k(\lambda).$$

- If $|\lambda| = n$, rescale axes by $\frac{1}{\sqrt{n}}$.
- $\frac{1}{\sqrt{n}}\varphi(\sqrt{n}t)$ is "almost surely very close to"

$$L(t) := \frac{\sqrt{12}}{\pi} \log\left(\frac{2}{1 + e^{-\frac{\pi t}{\sqrt{12}}}}\right).$$

Theorem (Dembo–Vershik–Zeitouni (1998))

For any $\varepsilon > 0$,

$$\lim_{n\to\infty} P_n\left(\sup_{t\geq 0}\left|\frac{1}{\sqrt{n}}\varphi(\sqrt{n}t) - L(t)\right| < \varepsilon\right) = 1.$$

Also, an explicit large deviation principle holds at the scaling of $\sqrt{n}$.

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Theorem (Yakubovich (2001))

For any fixed $0 < t_1 < \cdots < t_r$, *the vector*

$$\frac{1}{\sqrt{n}}\left(\varphi(t_1\sqrt{n}), \ldots, \varphi(t_r\sqrt{n})\right)$$

varies from $(L(t_1), \ldots, L(t_r))$ *like a* r*-dimensional Gaussian at the scaling* $n^{-\frac{1}{4}}$.

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Proposition (B., (2025+))

For fixed $0 < \delta < \frac{1}{4}$, *we have*

$$\limsup_{n\to\infty} n^{-\delta}\log P_n\left(\inf_{t\geq 0}\left|\frac{1}{\sqrt{n}}\varphi(\sqrt{n}t) - L(t)\right| > n^{-\frac{1}{4}+\delta}\right) < 0.$$

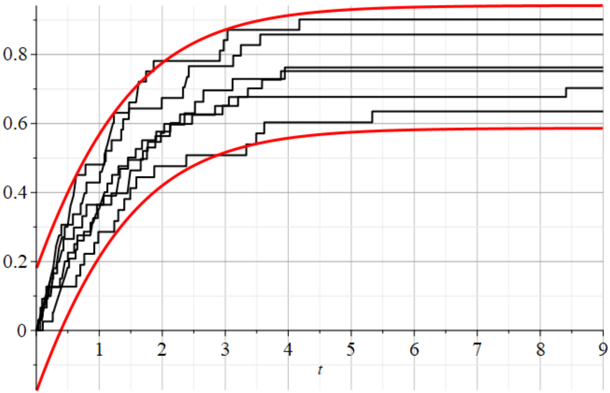

Figure: The black step functions are the renormalized shapes $\frac{1}{\sqrt{|\lambda|}}\phi_\lambda(\sqrt{|\lambda|}t)$ for six random distinct parts partitions λ of sizes 992, 1592, 1065, 1475, 910, and 1107, generated using a Boltzmann sampler with parameter $q = e^{-\frac{\pi}{\sqrt{12n}}}$ with $n = 1000$. In red are the curves $L(t) \pm n^{-\frac{1}{4}}$.

Large parts: sketch

- Observe that

$$\underbrace{\frac{1}{b\sqrt{n}}\sum_{a\sqrt{n}<k\leq b\sqrt{n}} X_k}_{\approx\frac{L(b)-L(a)}{b}} \leq \sum_{a\sqrt{n}<k\leq b\sqrt{n}} \frac{X_k}{k} \leq \underbrace{\frac{1}{a\sqrt{n}}\sum_{a\sqrt{n}<k\leq b\sqrt{n}} X_k}_{\approx\frac{L(b)-L(a)}{a}} .$$

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- Partition the interval (with appropriate scaling):

$$(K_n, n] = \bigcup_{j=1}^{J} (t_{j-1,n}\sqrt{n}, t_{j,n}\sqrt{n}].$$

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- Partition the interval (with appropriate scaling):

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- Limit shape proposition and careful analysis give (roughly)

$$\sum_{K_n<k\le n} \frac{2X_k}{k} \approx \sum_{j=1}^{J+1} \frac{2}{t_{j,n}}(L(t_{j,n}) - L(t_{j-1,n}))$$

$$\approx \log\left(\frac{\sqrt{3n}}{K_n}\right) - \gamma + o(1).$$

Boltzmann sampler

- The Boltzmann model is a family of probability distributions on **all** distinct parts partitions constructed from the generating function as

$$P(\lambda) := q^{|\lambda|} \prod_{k \geq 1} \frac{1}{1+q^k}, \qquad q \in (0,1).$$

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- Conditioned on the size, P coincides with the uniform measure.
- If $q = q_n = e^{-\frac{\pi}{\sqrt{12n}}}$ (the approximate saddle point of $q^{-n}\prod_{k\geq 1}(1+q^k)$), sampling X_k according to $\mathrm{Bernoulli}\left(\frac{q_n^k}{1+q_n^k}\right)$ yields a.s. partitions of size $n + O(n^{3/4})$ (Fristedt, 1993).

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- See, e.g., my October 2023 talk in this seminar for more details! :)

Unrestricted parts case

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- Distribution follows directly from work of Fristedt (1993) and Erdős–Lehner (1941).

Answer (see Kim–Kim, JCTA (2025))

For any $x \in \mathbb{R}$,

$$\lim_{n\to\infty} P_n\left(\frac{\pi}{\sqrt{6n}} S \le x\right) = 1 - \sum_{k\ge 1} (-1)^{k-1} e^{-k^2 x}.$$

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Remark

This is the **Kolmogorov distribution** *and arises in a number of places:*

1. *as the maximum height of the Brownian bridge process,*
2. *as the number of parts of partitions into squares (Goh–Hitczenko, 2006),*
3. *as the height of ordered, rooted trees on $n+1$ vertices (Renyi–Szekeres 1967, Stepanov 1969).*

Thanks for listening!

Our work:

- W. Bridges, "Distribution of the sum of reciprocal parts for distinct parts partitions," submitted. `arXiv:2503.03899`

Other references:

- K. Bringmann, B. Kim and E. Kim, Improved asymptotics for moments of reciprocal sums of partitions into distinct parts, preprint. `https://arxiv.org/pdf/2412.02534`
- B. Fristedt, "The structure of random large partitions of integers", Transactions of the American Mathematical Society **337** (1993), 703–735.
- B. Kim and E. Kim, "Distributions of reciprocal sums of parts in integer partitions", Journal of Combinatorial Theorey Series A **211** (2025).
- B. Schmuland, "Random harmonic series", American Mathematical Monthly **110** (2003).