Product-sum identities from symmetric cylindric and skew doubled shifted plane partitions

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- joint work with Ali Uncu (in preparation)

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$$\sum_{n \geq 0} \frac{q^{n^2+\epsilon n}}{\prod_{j=1}^{n}(1-q^j)} = \prod_{n \geq 0} \frac{1}{(1-q^{5n+1+\epsilon})(1-q^{5n+4-\epsilon})}.$$

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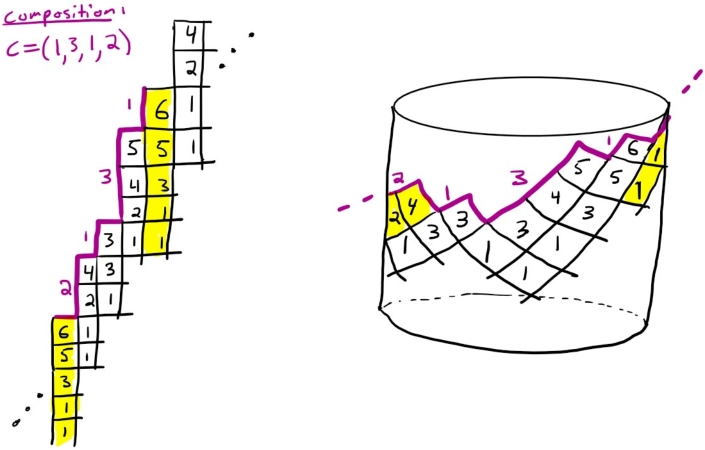
Our goal today:

- Extend Corteel–Welsh's idea to other similar structures.

Definitions and product sides

Gessel–Krattenthaler (1997, *Trans. AMS*), "Cylindric partitions"



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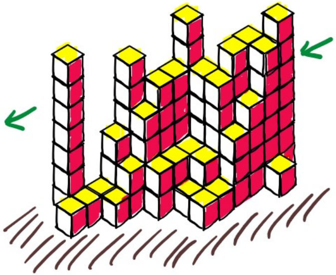
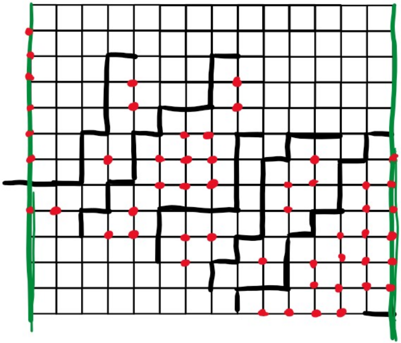




Notation: $\lambda \preceq \mu$ if $\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \dots$.

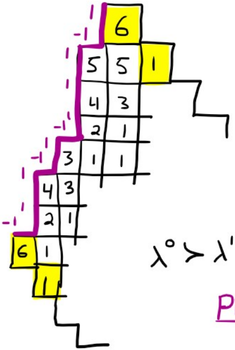
Definition

A *cylindric partition* $\lambda = (\lambda^0, \dots, \lambda^h)$ of width h and *profile* $(\delta_1, \dots, \delta_h) \in \{\pm 1\}^h$ is a sequence of $h+1$ partitions such that $\lambda^h = \lambda^0$ and

$$\begin{cases} \lambda^{j-1} \preceq \lambda^j & \text{if } \delta_j = 1, \\ \lambda^{j-1} \succeq \lambda^j & \text{if } \delta_j = -1. \end{cases}$$

The *size* is $|\lambda| = \sum_{j=0}^{h-1} |\lambda^j|$.

Example: a cylindric partition of size 33 and width 10, and profile $\delta = (-1, 1, 1, -1, 1, 1, 1, -1, 1)$,



$\lambda^0 = (6,1)$ $\quad \lambda^5 = (1)$

$\lambda^1 = (1)$ $\quad \lambda^6 = (2,1)$

$\lambda^2 = (2)$ $\quad \lambda^7 = (4,1)$

$\lambda^3 = (4,1)$ $\quad \lambda^8 = (5,3)$

$\lambda^4 = (3)$ $\quad \lambda^9 = (5)$

$\lambda^{10} = \lambda^0$

$$\lambda^0 \succ \lambda^1 \prec \lambda^2 \prec \lambda^4 \succ \lambda^5 \prec \lambda^6 \prec \lambda^7 \prec \lambda^8 \succ \lambda^9 \prec \lambda^{10}$$

PROFILE: $(-1, 1, 1, -1, 1, 1, 1, -1, 1)$

We use the standard q-Pochhammer notation:

$$\frac{1}{(a_1,\dots,a_r;q)_\infty} := \prod_{n\geq 0} \frac{1}{(1-a_1q^n)\cdots(1-a_rq^n)}.$$

Let $\mathcal{CP}_\delta$ be the set of cylindric partitions with profile δ, and let

$$\mathrm{CP}_\delta(q) := \sum_{\lambda\in\mathcal{CP}_\delta} q^{|\lambda|}.$$

Theorem (Borodin, 2007 Duke M. J., (Han–Xiong reformulation))

Let $\delta = (\delta_1, \ldots, \delta_h)$ *be a profile and let*

$$\begin{aligned} W_3(\delta) :=& \{j - i : 1 \le i < j \le h, \delta_i > \delta_j\} \\ & \cup \{h - (j - i) : 1 \le i < j \le h, \delta_i < \delta_j\}. \end{aligned}$$

Then

$$\mathrm{CP}_\delta(q) = \frac{1}{(q^h; q^h)_\infty} \prod_{k \in W_3(\delta)} \frac{1}{(q^k; q^h)_\infty}.$$

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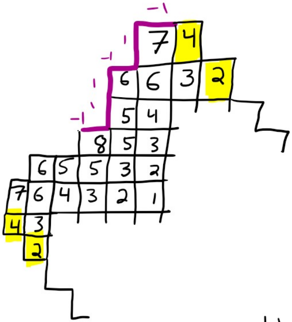
Then

$$\mathrm{CP}_\delta(q) = \frac{1}{(q^h; q^h)_\infty} \prod_{k \in W_3(\delta)} \frac{1}{(q^k; q^h)_\infty}.$$

Remark

This product is modular; that is, k *occurs as many times in* $W_3(\delta)$ *as* $h - k$. *This is far from obvious!*

Symmetric cylindric partitions are symmetric about the middle diagonal:



$\lambda^0 = (8,3,1)$

$\lambda^1 = (5,2)$

$\lambda^2 = (5,3)$

$\lambda^3 = (6,4)$

$\lambda^4 = (6)$

$\lambda^5 = (7,3)$

$\lambda^6 = (4,2)$

$\delta = (-1,1,1,-1,1,-1)$

$$|\lambda| = |\lambda^0| + |\lambda^6| + 2\sum_{j=1}^{5} |\lambda^j| = 100$$

Remark

We have $\mathrm{SCP}_\delta(q) = \mathrm{SCP}_{-\mathrm{rev}(\delta)}(q)$, *where* $\mathrm{rev}(\delta) = (\delta_h, \ldots, \delta_1)$.

Theorem (Han–Xiong*, 2019)

For a profile $\delta = (\delta_1, \dots, \delta_h)$, define the sets

$$\begin{aligned} W_6(\delta) :=& \{2h\} \cup \{2i-1 : \delta_i = -1\} \cup \{2h-2i+1 : \delta_i = 1\}, \\ W_7(\delta) :=& \{2j+2i-2 : 1 \le i < j \le h, \delta_i = \delta_j = -1\} \\ & \cup \{4h+2-2j-2i : 1 \le i < j \le h, \delta_i = \delta_j = 1\} \\ & \cup \{4h+2i-2j : 1 \le i < j \le h, \delta_i < \delta_j\} \\ & \cup \{2j-2i : 1 \le i < j \le h, \delta_i > \delta_j\}\}. \end{aligned}$$

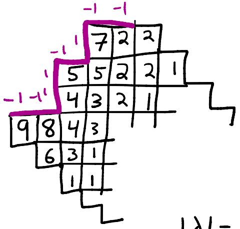
Then

$$\mathrm{SCP}_\delta(q) := \sum_{\lambda \in \mathcal{SCP}_\delta} q^{|\lambda|} = \prod_{\substack{k \in W_6(\delta) \\ \ell \in W_7(\delta)}} \frac{1}{(q^k; q^{2h})_\infty (q^\ell; q^{4h})_\infty}.$$

Remark

Unlike for $\mathrm{CP}_\delta(q)$, this product is not modular in general.

(Han–Xiong, 2019) For *skew doubled shifted plane partitions* (DSPP), the partitions λ^0 and λ^h need not be identical:



$\delta = (-1, -1, 1, 1, -1, 1, -1, -1)$

$\lambda^0 = (9,6,1)$ $\quad \lambda^4 = (5,3)$ $\quad \lambda^8 = (2,1)$

$\lambda^1 = (8,3,1)$ $\quad \lambda^5 = (5,2)$

$\lambda^2 = (4,1)$ $\quad \lambda^6 = (7,2,1)$

$\lambda^3 = (4,3)$ $\quad \lambda^7 = (2,2)$

$$|\lambda| = \sum_{j=0}^{8} |\lambda^j| = 72$$

Remark

We have $\mathrm{DSPP}_\delta(q) = \mathrm{DSPP}_{-\mathrm{rev}(\delta)}(q)$, *where* $\mathrm{rev}(\delta) = (\delta_h, \ldots, \delta_1)$.

Theorem (Han–Xiong, 2019)

For a profile δ of width h, define the sets

$$\begin{aligned} W_1(\delta) :=& \{h+1\} \cup \{i : \delta_i = -1\} \cup \{h+1-i : \delta_i = 1\}, \\ W_2(\delta) :=& \{i+j : 1 \leq i < j \leq h,\ \delta_i = \delta_j = -1\} \\ & \cup \{2h+2-i-j : 1 \leq i < j \leq h,\ \delta_i = \delta_j = 1\} \\ & \cup \{2h+2-(j-i) : 1 \leq i < j \leq h,\ \delta_i < \delta_j\} \\ & \cup \{j-i : 1 \leq i < j \leq h,\ \delta_i > \delta_j\}. \end{aligned}$$

Then

$$\mathrm{DSPP}_\delta(q) := \sum_{\lambda \in \mathcal{DSPP}_\delta} q^{|\lambda|} = \prod_{\substack{k \in W_1(\delta) \\ \ell \in W_2(\delta)}} \frac{1}{(q^k; q^{h+1})_\infty (q^\ell; q^{2h+2})_\infty}.$$

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- As a corollary of their work, one can prove product generating functions when size is counted with any *nonnegative weights*.

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- As a corollary of their work, one can prove product generating functions when size is counted with any *nonnegative weights*.

Definition

For $a = (a_0, a_1, \ldots, a_h) \in \mathbb{R}_+^{h+1}$, define $A_j := \sum_{i=0}^{j-1} a_i$. For $\lambda \in \mathcal{DSPP}_\delta$, define a *weighted size*

$$|\lambda|_a := \sum_{i=0}^{h} a_i |\lambda^i|.$$

- This allows us to get *even more* "product sides".

Proposition (B.-Uncu)

For a profile δ of width h, define

$$\begin{aligned}
W_1^a(\delta) :=& \{A_{h+1}\} \cup \{A_j : \delta_j = -1\} \cup \{A_{h+1} - A_j : \delta_j = 1\},\\
W_2^a(\delta) :=& \{A_i + A_j : 1 \leq i < j \leq h,\ \delta_i = \delta_j = -1\}\\
& \cup \{2A_{h+1} - A_i - A_j : 1 \leq i < j \leq h,\ \delta_i = \delta_j = 1\}\\
& \cup \{2A_{h+1} - (A_j - A_i) : 1 \leq i < j \leq h,\ \delta_i < \delta_j\}\\
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\end{aligned}$$

Then

$$\mathrm{DSPP}_\delta^a(q) := \sum_{\lambda \in \mathcal{DSPP}_\delta} q^{|\lambda|_a} = \prod_{\substack{k \in W_1^a(\delta)\\ \ell \in W_2^a(\delta)}} \frac{1}{(q^k; q^{A_{h+1}})_\infty (q^\ell; q^{2A_{h+1}})_\infty}.$$

Remark

For $a = (1, 2, 2, \ldots, 2, 1)$, we have $\mathrm{SCP}_\delta(q) = \mathrm{DSPP}_\delta^a(q)$.

For cylindric partitions of width h, we define $|\lambda|_a := \sum_{i=0}^{h-1} a_i|\lambda^i|$, so that λ^h is counted with λ^0.

Proposition (B.-Uncu)

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Then

$$\mathrm{CP}_\delta^a(q) := \sum_{\lambda \in \mathcal{CP}_\delta} q^{|\lambda|_a} = \prod_{k \in W_3^a(\delta)} \frac{1}{(q^{A_k}; q^{A_h})_\infty}.$$

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- Capparelli $\#1$:

$$\frac{1}{(q^2, q^3, q^9, q^{10}; q^{12})_\infty} = (q^{12}; q^{12})_\infty \mathrm{CP}^{(1,2,7,2)}_{(-1,1,1,-1)}(q)$$

- Kanade–Russell I_1:

$$\frac{1}{(q, q^3, q^6, q^8; q^9)_\infty} = (q^9; q^9)_\infty \mathrm{CP}^{(1,2,1,5)}_{(1,1,-1,-1)}(q)$$

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- For $0 < b_1 \leq b_2 \leq \cdots \leq b_{r+1}$, we have

$$\frac{1}{(q^{b_1}, \ldots q^{b_r}; q^{b_{r+1}})_\infty} = (q^{b_{r+1}}; q^{b_{r+1}})_\infty \mathrm{CP}^{(b_1, b_2 - b_1, \ldots, b_{r+1} - b_r)}_{(-1,-1,\ldots,-1,1)}(q).$$

But this profile does not lead to interesting recurrences/sum-sides...

Recurrences and sum sides

Toy example:

- Let $P(z;q) := \sum_{\lambda \in \mathcal{P}} z^{\ell(\lambda)} q^{|\lambda|}$, where the sum runs over all integer partitions and $\ell(\lambda)$ is the size of the *largest part*.

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$$P(z;q) = \frac{P(zq;q)}{1-zq} = \sum_{m \geq 0} z^m q^m P(zq;q) = \sum_{\substack{m \geq 0 \\ \lambda \in \mathcal{P}}} z^{m+\ell(\lambda)} q^{m+\ell(\lambda)+|\lambda|}.$$

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- This leads to the product-sum identity:

$$P(z;q) = \sum_{n\geq 0} \frac{z^n q^n}{(q;q)_n} = \frac{1}{(zq;q)_\infty}.$$

Let powers of z in $\mathrm{CP}_\delta(z;q)$ count the size of the largest square in a cylindric partition.

Proposition (Corteel–Welsh, 2019 Annals of Comb.)

Let $\delta = (\delta_1, \dots, \delta_h)$ be a profile. For convenience define $\delta_{h+1} := \delta_1$. Define

$$I_\delta := \{1 \leq j \leq h : (\delta_j, \delta_{j+1}) = (1, -1)\}.$$

For a subset $\emptyset \subsetneq J \subseteq I_\delta$, define a new profile $\sigma_J(\delta)$ by swapping the signs of (δ_j, δ_{j+1}) for $j \in J$. Then

$$\mathrm{CP}_\delta(z;q) = \sum_{\emptyset \subsetneq J \subseteq I_\delta} (-1)^{|J|-1} \frac{\mathrm{CP}_{\sigma_J(\delta)}\left(zq^{|J|};q\right)}{1 - zq^{|J|}}.$$

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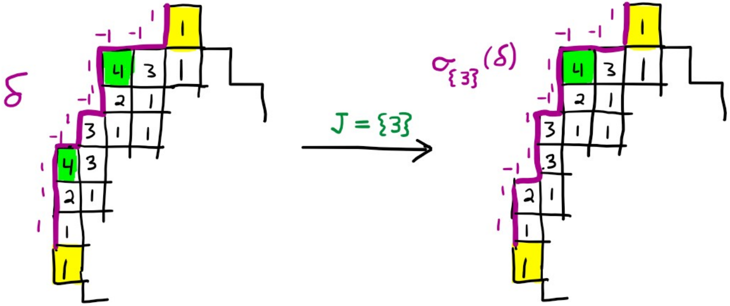
Proof idea: Given a cylindric partition of profile δ, the set I_δ locates possible squares for the largest part. Removing some subset J of largest parts changes the profile into $\sigma_J(I_\delta)$. The (-1)'s indicate inclusion-exclusion.

Example:



Choose $J \subseteq I_\delta = \{3, 8\}$

Similarly,

Proposition (B.-Uncu)

$$\mathrm{CP}^{a}_{\delta}(z;q) = \sum_{\emptyset \subsetneq J \subseteq I_\delta} (-1)^{|J|-1} \frac{\mathrm{CP}^{a}_{\sigma_J(\delta)}\left(zq^{\sum_{i\in J} a_i};q\right)}{1 - zq^{\sum_{i\in J} a_i}},$$

and

$$\mathrm{DSPP}^{a}_{\delta}(z;q) = \sum_{\emptyset \subsetneq J \subseteq I_\delta} (-1)^{|J|-1} \frac{\mathrm{DSPP}^{a}_{\sigma_J(\delta)}\left(zq^{\sum_{i\in J} a_i};q\right)}{1 - zq^{\sum_{i\in J} a_i}}.$$

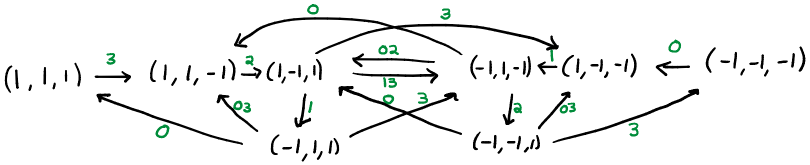
For DSPP the σ_J have a slightly different meaning!

Remark

Note that the width is preserved by σ. In fact, for DSPP there is one system of 2^h recurrences for width h. For CP, there are h separate systems of $\binom{h}{j}$ recurrences for width h and profiles with j (-1)'s.

Example identities

For DSPP with standard weight and width 3, the directed graph below describes the system of recurrences.



$$\delta \xrightarrow{J} \varepsilon \quad \text{when} \quad \varepsilon = \sigma_J(\delta)$$

Solving the recurrences and employing the known product generating function:

Corollary ("Göllnitz–Gordon" and "Little Göllnitz" Identities)

We have

$$\sum_{n\geq 0} \frac{\left(-q;q^2\right)_n}{(q^2;q^2)_n} q^{n^2+2n} = \frac{1}{(q^3,q^4,q^5;q^8)_\infty},$$

$$\sum_{n\geq 0} \frac{\left(-q;q^2\right)_n}{(q^2;q^2)_n} q^{n^2+n} = \frac{1}{(q^3;q^4)_\infty\,(q^2;q^8)_\infty},$$

$$\sum_{n\geq 0} \frac{\left(-q;q^2\right)_n}{(q^2;q^2)_n} q^{n^2} = \frac{1}{(q,q^4,q^7;q^8)_\infty},$$

$$\sum_{n\geq 0} \frac{\left(-q^{-1};q^2\right)_n}{(q^2;q^2)_n} q^{n^2+n} = \frac{1}{(q;q^4)_\infty\,(q^6;q^8)_\infty}.$$

4 identities instead of $8 = 2^3$ because of $\delta \mapsto -\mathrm{rev}(\delta)$ symmetry.

Solving the width 2 symmetric cylindric case (equivalently DSPP with weight $a = (1, 2, 1)$) leads to

Theorem (B.-Uncu)

$$\sum_{n\geq 0}(-1)^n q^{4n^2}\frac{(q^2,-q^4;q^4)_n}{(q^4;q^4)_{2n}}\left(1-\frac{q^{4n+1}z}{(1+q^{4n+2})}\right)z^{2n} = (zq,-zq^3;q^4)_\infty.$$

Solving the width 3 symmetric cylindric case (equivalently DSPP with weight $a = (1, 2, 2, 1)$) leads, for example, to

Theorem (B.-Uncu)

$$\sum_{m,n\geq 0}(-1)^m \frac{q^{3\lceil n/2\rceil^2+3\lfloor n/2\rfloor(\lfloor n/2\rfloor+1)-3m(m+1)}}{(q^3;q^3)_n}\begin{bmatrix} n \\ 2m \end{bmatrix}_{q^3}$$
$$\times(-q,q^3,-q^5;q^6)_m = \frac{(q^4,q^8;q^{12})_\infty}{(q^6;q^{12})_\infty}.$$

More applications

"Frank Schmidt-type problems" (cf. Andrews–Paule, Andrews' talk on Sept. 23 2021, see also Uncu, 2018)

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1. $\frac{1}{(q;q)_\infty} = \sum_{\lambda \in \mathcal{D}} q^{\lambda_1+\lambda_3+\lambda_5+\dots}$
2. $\frac{1}{(q;q)_\infty^2} = \sum_{\lambda \in \mathcal{P}} q^{\lambda_1+\lambda_3+\lambda_5+\dots}$

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2. $\frac{1}{(q;q)_\infty^2} = \sum_{\lambda\in\mathcal{P}} q^{\lambda_1+\lambda_3+\lambda_5+\dots}$
3. $\frac{(-q;q)_\infty}{(q;q)_\infty^3} = \sum_{\lambda\in\mathrm{DIAMOND}} q^{\lambda_1+\lambda_4+\lambda_7+\dots}$, where

$\lambda \in \mathrm{DIAMOND}$ if

$$\begin{array}{ccccccc}
\lambda_1 & \geq & \lambda_2 & & & & \\
\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & & & \\
\lambda_3 & \geq & \lambda_4 & \geq & \lambda_5 & & \\
 & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \\
 & & \lambda_6 & \geq & \lambda_7 & & \dots \\
 & & & & \vdots & &
\end{array}$$

Our weighted product formulas confirm these identities; for example,

$$\sum_{\lambda \in \mathrm{DIAMOND}} q^{\lambda_1+\lambda_4+\lambda_7+\dots} = \mathrm{DSPP}^{(0,1,0)}_{(1,-1)}(q)$$

$$= \frac{1}{(q;q)^3_\infty (q;q^2)_\infty}$$

$$= \frac{(-q;q)_\infty}{(q;q)^3_\infty}$$

The second identity may be verified similarly.

Thanks for listening!

Note: Ali Uncu will add $\mathrm{CP}^a_\delta(z;q)$ and $\mathrm{DSPP}^a_\delta(z;q)$ recurrences to the qfunctions package in Mathematica

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