Methods in asymptotic statistics for partitions

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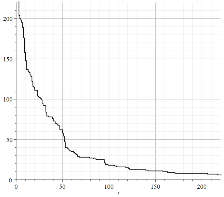
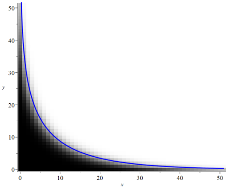
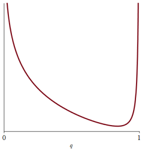
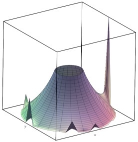
October 12, 2023









International Conference on Modular Forms and q-Series

University of Cologne, **March 11-15, 2024**

Organizers: Kathrin Bringmann, Walter Bridges, Johann Franke

http://www.mi.uni-koeln.de/Bringmann/Konferenz2024.html

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Why care about statistics for partitions?

- Structure of the symmetric group S_n.
- Statistical mechanics of ideal gas (Boltzmann, Temperley, Vershik...)
- Partitions serve as a prototype for other *logarithmic* combinatorial structures (Arratia–Tavaré 2000).

Primary Goal. Overview of the Boltzmann model; sketch distributions of small parts, large parts, Young diagrams, asymptotic formula for $p(n)$, etc.

Secondary Goal. Discuss my contributions:

- (with K. Bringmann), *Limit shapes for Andrews–Gordon partitions,* on-going work
- (with K. Bringmann), *Statistics for unimodal sequences*, Adv. in Math. **401** (2022)
- *Limit shapes for unimodal sequences*, Int. J. Number Theory **19** (2023)

Part 1: Boltzmann models for partitions

Question

How many partitions of n contain a 1?

$$\#\{\lambda \vdash n : 1 \in \lambda\} = p(n-1)$$

Question

What is the **probability** *that a partition of n contains a 1?*

$$\frac{\#\{\lambda \vdash n : 1 \in \lambda\}}{p(n)} = \frac{p(n-1)}{p(n)} \overset{n\to\infty}{\to} 1.$$

Answer

As $n \to \infty$, 100% of partitions of n contain a 1.

Question

What is the **probability** *that a partition of n contains two 1s, one 2 and one 5?*

$$\frac{\#\{\lambda \vdash n : 1+1+2+5 \in \lambda\}}{p(n)} = \frac{p(n-1-1-2-5)}{p(n)} \overset{n\to\infty}{\to} 1.$$

Answer

As $n \to \infty$, 100% of partitions of n contain two 1s, one 2 and one 5.

Question

What is the **probability** *that a partition of n contains n 1s?*

$$\frac{\#\{\lambda \vdash n : \overbrace{1 + \cdots + 1}^{n \text{ times}} \in \lambda\}}{p(n)} = \frac{1}{p(n)} \overset{n\to\infty}{\to} 0.$$

Answer

As $n \to \infty$, 0% of partitions of n contain n 1s.

Define $M_1(\lambda) :=$ *multiplicity* of 1s in the partition λ,
P_n - *uniform measure* on $\{\lambda \vdash n\}$.

We have seen:

$$P_n(M_1 \geq 1) \to 1, \qquad P_n(M_1 \geq n) = P_n\left(\frac{1}{n} M_1 \geq 1\right) \to 0.$$

Question

What scaling of M_1 yields a non-trivial distribution?

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Theorem (Erdős–Turán, Fristedt, ...)

Let $A := \frac{\sqrt{6}}{\pi}$. As $n \to \infty$, M_1 obeys an exponential distribution at the scale of $\sqrt{n}$:

$$P_n\left(\frac{1}{A\sqrt{n}} M_1 \leq x\right) \to 1 - e^{-x}.$$

Proof.

Use $p(m) \sim \frac{1}{4\sqrt{3}m} e^{\pi\sqrt{\frac{2m}{3}}}$ in

$$P_n\left(\frac{1}{A\sqrt{n}} M_1 \leq x\right) \stackrel{*}{=} 1 - \frac{p\left(n - A\sqrt{n}x\right)}{p(n)}. \qquad \square$$

Other multiplicities M_k of small parts are similar.

Theorem (Erdős–Turán, Fristedt, . . .)

Let $k = o(\sqrt{n})$. As $n \to \infty$, M_k obeys an exponential distribution at the scale of $\frac{\sqrt{n}}{k}$:

$$P_n\left(\frac{k}{A\sqrt{n}} M_k \leq x\right) \to 1 - e^{-x}.$$

What about $L_1(\lambda) :=$ the largest part of λ?

By inclusion-exclusion:

$$P_n(L_1 \leq k) = \frac{1}{p(n)} \left(p(n) - \sum_{r_1 \geq 1} p\left(n - (k + r_1)\right) \right.$$
$$+ \sum_{r_2 > r_1 \geq 1} p\left(n - (k + r_1) - (k + r_2)\right)$$
$$\left. - \dots \right).$$

The scaling that yields a non-trivial distribution is

$$\frac{L_1 - A\sqrt{n} \log\left(A\sqrt{n}\right)}{A\sqrt{n}}.$$

What about $L_1(\lambda) :=$ the largest part of λ?

Theorem (Erdős–Lehner 1941)

$$P_n\left(\frac{L_1 - A\sqrt{n}\log\left(A\sqrt{n}\right)}{A\sqrt{n}} \leq x\right) \to e^{-e^{-x}}.$$

Interpretation

Typically, the largest part is about $A\sqrt{n}\log(A\sqrt{n})$ *and the difference varies by an* **extreme value distribution**.

Earlier methods (Erdős, Szalay, Szekeres, Turán, ...):

$$\text{combinatorics/recurrences} + \text{asymptotics}$$

An alternative method (Fristedt, Pittel, Vershik, ... 1990s) is to use the **Boltzmann model***, which has the following nice properties.

* also called Fristedt's conditioning device, the Arratia–Tavaré principle, etc.

- Avoids intricate combinatorics, recurrences, etc., ... and generalizes to situations which lack nice combinatorics (e.g. unimodal sequences).
- Offers heuristic insight (e.g. for the asymptotic formula for $p(n)$).
- Allows families of distributions to be derived simultaneously.
- Can be used to quickly generate random partitions.

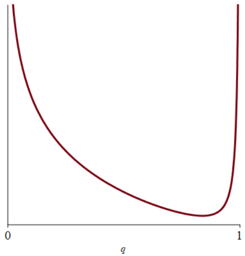
Key idea 1: product generating function

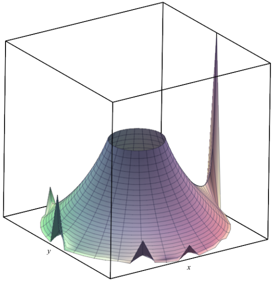
Sequence of multiplicities $(M_1, M_2, \dots)$, which completely determines a partition, is generated *independently* by

$$P(q) := \sum_{n \geq 0} p(n) q^n = \prod_{k \geq 1} \frac{1}{1 - q^k}.$$

Key idea 2: saddle-point bound/approximation

Note that $p(n) \le q^{-n} P(q)$ for $q \in (0,1)$.





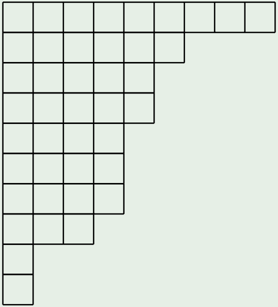
Choosing $q = q_n := e^{-\frac{1}{A\sqrt{n}}}$, the RHS is **close enough** to $p(n)$:

$$\log p(n) \sim \log q_n^{-n} P(q_n), \quad \text{in particular} \quad \frac{p(n)}{q_n^{-n} P(q_n)} \sim \textbf{Polynomial}.$$

Notation:

Example

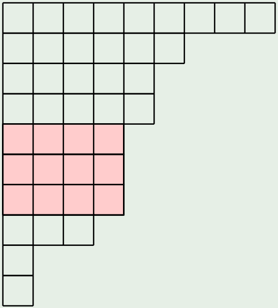
$$\lambda : 9+6+5+5+4+4+4+2+1+1$$



Notation: multiplicity

Example

$$\lambda : 9+6+5+5+4+4+4+2+1+1$$

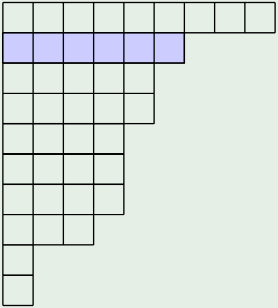


$$M_4(\lambda) = 3$$

Notation: largest parts

Example

$$\lambda : 9 + 6 + 5 + 5 + 4 + 4 + 4 + 2 + 1 + 1$$



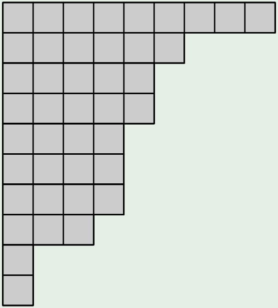
$$L_2(\lambda) = 6$$

$L_1 = \sup\{k : M_k > 0\}$ has same distribution as $\sum_{k \geq 1} M_k$.

Notation: size

Example

$$\lambda : 9 + 6 + 5 + 5 + 4 + 4 + 4 + 2 + 1 + 1$$



$$S(\lambda) = 41$$

Note $S = \sum_{k \geq 1} k M_k$.

Recall $P(q) = \prod_{k\geq 1}(1-q^k)^{-1}$.

Boltzmann model: For $q \in (0,1)$, define

$$\boldsymbol{P}_q(\lambda) := \frac{q^{S(\lambda)}}{P(q)}.$$

- We get a probability measure on **all** partitions.
- It is uniform when restricted to partitions of n:

$$\boldsymbol{P}_q(\cdot|S=n) = P_n(\cdot)$$

- By **Key Idea 1**, M_k are **independent** geometric:

$$\boldsymbol{P}_q(M_k = \ell) = q^{k\ell}(1-q^k).$$

Transfer Principle

By **Key Idea 2** (saddle-point approximation), distributions under P_n coincide with those under $\boldsymbol{P}_{q_n}$, as $n \to \infty$, for *most* statistics.

d_{TV} - total variation metric on probability measures.

Proposition (Fristedt, Trans. AMS, 1993)

Suppose X is a random variable determined by $\{M_k : k \leq a_n \text{ or } k \geq b_n\}$. If $a_n = o(\sqrt{n})$ and $b_n = \omega(\sqrt{n})$, then

$$\lim_{n\to\infty} d_{TV}\left(P_n(X^{-1}), \boldsymbol{P}_{q_n}(X^{-1})\right) = 0.$$

Question

What is the joint distribution of $(M_1, \ldots, M_{k_n})$ under P_n?

Theorem (Fristedt 1993, *Trans. AMS*)

For $k_n = o\left(n^{\frac{1}{4}}\right)$. Then

$$P_n\left(\frac{kM_k}{A\sqrt{n}} \leq x_k, 1 \leq k \leq k_n\right) \sim \prod_{k=1}^{k_n} \left(1 - e^{-x_k}\right)$$

Answer

The multiplicities of $k = o(n^{1/4})$, rescaled by $\frac{\sqrt{n}}{k}$, behave as **independent exponential** *random variables.*

Question

How is the size S distributed under $\boldsymbol{P}_{q_n}$?

Recall $S = \sum_{k\geq 1} \underbrace{kM_k}_{\text{independent!}}$. Here,

$$\text{Mean:} \qquad \mathrm{E}_{q_n}(S) = \sum_{k\geq 1} \frac{kq_n^k}{1-q_n^k} \sim n$$

$$\text{Variance:} \qquad \mathrm{Var}_{q_n}(S) = \sigma_n^2 = \sum_{k\geq 1} \frac{k^2 q_n^k}{(1-q_n^k)^2} \sim 2An^{\frac{3}{2}}.$$

Remark

Solving for $q \in (0,1)$ in $\mathrm{E}_q(S) \sim n$ is **equivalent** *to finding the saddle-point of $q^{-n}P(q)$.*

Question

How is the size S distributed under $\boldsymbol{P}_{q_n}$?

(Fristedt, 1993): S satisfies a central limit theorem under $\boldsymbol{P}_{q_n}$:

$$\boldsymbol{P}_{q_n}\left(\frac{S-n}{\sqrt{2A}n^{3/4}} \leq x\right) \sim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt.$$

This leads to a **heuristic** derivation of the asymptotic formula for $p(n)$.

$$\boldsymbol{P}_q\left(S=n\right)=\sum_{\lambda\vdash n}\frac{q^{S(\lambda)}}{P(q)}=p(n)\frac{q^n}{P(q)}.$$

Equivalently, saddle-point bound is repaired:

$$p(n)=\boldsymbol{P}_q\left(S=n\right)q^{-n}P(q).$$

Recalling the normal distribution, we expect a **local central limit theorem**:

$$\begin{aligned}\boldsymbol{P}_{q_n}\left(S=n\right)&=\boldsymbol{P}_{q_n}\left(-1<S-n<1\right)\\&=\boldsymbol{P}_{q_n}\left(-\frac{1}{\sqrt{2A}n^{3/4}}<\frac{S-n}{\sqrt{2A}n^{3/4}}<\frac{1}{\sqrt{2A}n^{3/4}}\right)\\&\overset{??}{\sim}\frac{1}{\sqrt{2\pi}}\int_{-1/\sqrt{2A}n^{3/4}}^{1/\sqrt{2A}n^{3/4}}e^{-\frac{t^2}{2}}\,dt\\&\sim\frac{2}{\sqrt{2A}n^{3/4}}.\end{aligned}$$

Substituting

$$\boldsymbol{P}_{q_n}(S=n) \sim \frac{2}{\sqrt{2A}n^{3/4}},$$

and the asymptotic for $q_n^{-n}P(q_n)$ into

$$p(n) = \boldsymbol{P}_q(S=n)\, q^{-n} P(q),$$

one recovers

Theorem (Hardy–Ramanujan)

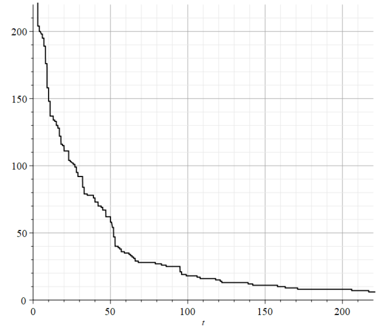
$$p(n) \sim \frac{1}{4\sqrt{3}n} q^{\pi\sqrt{\frac{2n}{3}}}.$$

Application: sampling large partitions

1. Generate **independent** sample multiplicities according to $M_k \sim \text{Geometric}(1 - q_n^k)$
2. The size $S = \sum_{k \geq 1} kM_k$ is very likely $n + O(n^{3/4})$.

Example

Let $n = 10,000$, so $n^{3/4} = 1000$. I get the following partition of 9599.

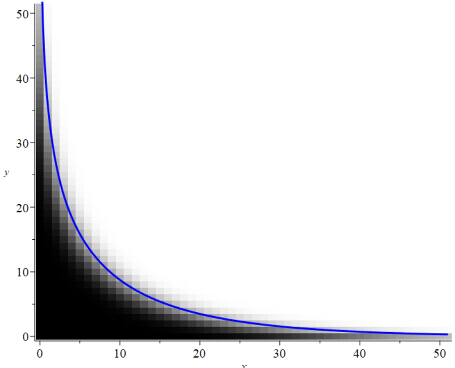


Limit shapes

Question

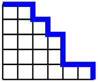
What are the typical shapes of diagrams of partitions of n?

Figure: Density plot of $\{\lambda \vdash 300\}$.



$\widetilde{\varphi}(\lambda)$ - renormalized shape; rescale by $\frac{1}{\sqrt{n}}$

$\frac{5}{\sqrt{19}}$



$\frac{6}{\sqrt{19}}$

Total Area $= 1$

Question

What are the typical shapes of diagrams of partitions of n?

Theorem (Dembo–Vershik–Zeitouni 1998)

Let N_ϵ be an ϵ-strip around $e^{-\frac{x}{A}} + e^{-\frac{y}{A}} = 1$.

$$\lim_{n\to\infty} P_n\left(\widetilde{\varphi}(\lambda) \subset N_\epsilon\right) = 1.$$

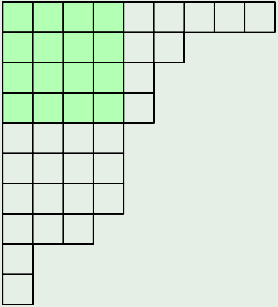
Also, an explicit large deviation principle holds.

Conjectured/heuristically derived by Temperley (1952), Szalay-Turán (1977), Vershik (1996), ...

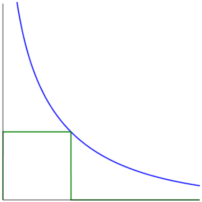
Application: Durfee square side length

Example

$$\lambda : 9 + 6 + 5 + 5 + 4 + 4 + 4 + 2 + 1 + 1$$



$$\text{Durfee}(\lambda) = 4$$



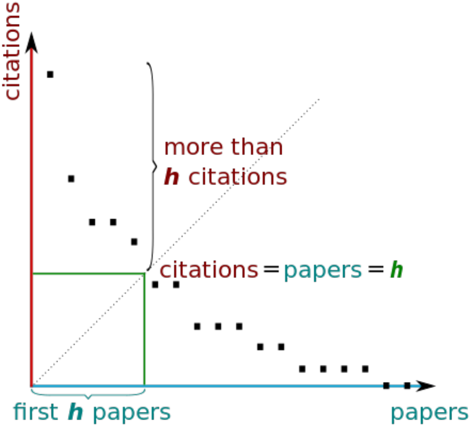
$$e^{-\frac{x}{A}} + e^{-\frac{x}{A}} = 1$$
$$\implies x = A\log(2).$$

Proposition

For any $\epsilon > 0$,

$$P_n\left(\left|\frac{Durfee}{\sqrt{n}} - A\log(2)\right| < \epsilon\right) \to 1.$$

(Canfield–Corteel–Savage, 1998) away from this mean, the Durfee square is normally distributed.



The h-index is an **information-less** statistic.

C. Krattenthaler, *Was der h-Index wirklich aussagt*, Mitt. Dtsch. Math.-Ver. 29 (2021), 124–128

But why can we use the Boltzmann model here? Limit shape depends on parts of size roughly $\sqrt{n}$.

Proposition (Fristedt, Trans. AMS, 1993)

Suppose X is a random variable determined by $\{M_k : k \leq a_n \text{ or } k \geq b_n\}$. If $a_n = o(\sqrt{n})$ and $b_n = \omega(\sqrt{n})$, then

$$\lim_{n\to\infty} d_{TV}\left(P_n(X^{-1}), \boldsymbol{P}_{q_n}(X^{-1})\right) = 0.$$

Also, largest part depends on all part sizes: $L_1 = \sup\{k : M_k > 0\}$. Why can we use the Boltzmann model here?

"Exponentially-small Principle"

Lemma

If $\boldsymbol{P}_{q_n}(Event)$ is exponentially small, then

$$P_n(Event) \to 0.$$

Proof.

$$\begin{aligned} P_n(\text{Event}) &= \boldsymbol{P}_{q_n}(\text{Event}|S = n) \\ &= \frac{\boldsymbol{P}_{q_n}(\text{Event} \cap S = n)}{\boldsymbol{P}_{q_n}(S = n)} \\ &\leq \frac{\boldsymbol{P}_{q_n}(\text{Event})}{\boldsymbol{P}_{q_n}(S = n)}. \end{aligned}$$

But $\boldsymbol{P}_{q_n}(S = n) = p(n) q_n^n P(q_n)^{-1}$ decays only **polynomially**. $\square$

Application: Largest part

Lemma

If $\boldsymbol{P}_{q_n}(Event)$ is exponentially small, then

$$P_n(Event) \to 0.$$

There exists $b_n = \omega(\sqrt{n})$ such that

$$\begin{aligned} &\boldsymbol{P}_{q_n}(M_k = 0, k \geq b_n) \text{ exp. small} \\ \implies\ & P_n(M_k = 0, k \geq b_n) \to 0 \\ \implies\ & P_n(M_k > 0 \text{ for some } k \geq b_n) \to 1. \end{aligned}$$

Thus, $L_1 = \sup\{k \geq 1 : M_k > 0\}$ has the same limiting distribution under P_n as $\sup\{k \geq b_n : M_k > 0\}$. Can now apply the Fristedt's Transfer Principle.

Question

How is $(L_1, \ldots, L_{t_n})$ distributed under P_n as $n \to \infty$?

Theorem (Fristedt 1993, *Trans. AMS*)

Let $t_n = o(n^{1/4})$ and $v_1 \geq \cdots \geq v_{t_n}$. Then

$$P_n\left(\frac{L_t - A\sqrt{n}\log\left(A\sqrt{n}\right)}{A\sqrt{n}} \leq v_t, 1 \leq t \leq t_n\right)$$

$$\sim \int_{-\infty}^{v_1} \cdots \int_{-\infty}^{v_{t_n}} e^{-u_1 - \cdots - u_{t_n} - e^{-u_{t_n}}}\, du_{t_n} \cdots du_1.$$

Answer

Typically, the largest t_n parts are all about $A\sqrt{n}\log(A\sqrt{n})$ and away from this mean behave as a **Markov chain**.

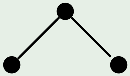
Unexpected application?

Definition

A partition λ is **graphical** if its parts are the vertex degrees of a simple graph.

Example

$\lambda = 2 + 1 + 1$ is the only graphical partition of 4.



Conjecture (Wilf (1982))

$\lim_{n\to\infty} P_n(\lambda \textit{ is graphical}) = 0.$

Proposition (Erdős–Gallai (1960))

λ is graphical if and only if

$$\sum_{t=1}^{d} L_t(\overline{\lambda}) \geq \sum_{t=1}^{d} (L_t(\lambda) + t) \qquad \textit{for } d \leq \textit{Durfee}(\lambda).$$

Theorem (Pittel (1997), *JCTA*)

Wilf's conjecture is true.

Theorem (Melczer–Michelen–Mukherjee (2020), *IMRN*)

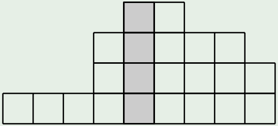
For some $C > 0$, we have $P_n(\lambda \textit{ is graphical}) \leq Cn^{-.003297210314}$.

Part 2: Boltzmann models for unimodal sequences

Notation: peaks

Example

$$\lambda : 1 + 1 + 1 + 3 + \underbrace{\bar{4}}_{\text{peak}} + 4 + 3 + 3 + 2$$

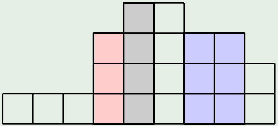


$$PK(\lambda) = 4$$

Notation: multiplicites

Example

$$\lambda : 1 + 1 + 1 + 3 + \overline{4} + 4 + 3 + 3 + 2$$

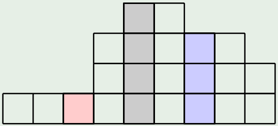


$$M_3^{[\ell]}(\lambda) = 1 \qquad M_3^{[r]}(\lambda) = 2$$

Notation: largest parts

Example

$$\lambda : 1 + 1 + 1 + 3 + \overline{4} + 4 + 3 + 3 + 2$$



$$\color{red}{L_2^{[\ell]}(\lambda) = 1} \qquad \color{blue}{L_2^{[r]}(\lambda) = 3}$$

Key Ideas

$$U(q) = \sum_{n \geq 0} u(n) q^n = \sum_{m \geq 0} \underbrace{q^m}_{\text{peak}} \prod_{k \leq m} \underbrace{\frac{1}{(1-q^k)^2}}_{\text{left/right parts}}$$

$U(q)$ is not a product, and $M_k^{[\ell]}, M_k^{[r]}$ are **not** independent under the naïve Boltzmann model:

$$\boldsymbol{P}_q(\lambda) := \frac{q^{S(\lambda)}}{U(q)}, \qquad q \in (0,1),$$

nor is the RHS very tractable. But **conditioning** on the peak we have a product generating function,

$$q^m \prod_{k \leq m} \frac{1}{(1-q^k)^m},$$

and recover independence. We then apply the Boltzmann model **uniformly** for m in the contributing range.

Transfer Principle

Let $B = \frac{\pi}{\sqrt{3}}$, $q_n = e^{-1/B\sqrt{n}}$ and

$$Q_{q,m}(\cdot) := Q_q\left(\cdot|\mathrm{PK} = m\right) \qquad P_{n,m}(\cdot) := P_n\left(\cdot|\mathrm{PK} = m\right),$$

Proposition (B.–Bringmann, Adv. Math. 2022)

Suppose $X : \mathcal{U} \to \mathbb{R}^{d_n}$ is a random variable determined by $\{M_k^{[j]}\}_{k \in K_{n,m}, j \in \{\ell, r\}}$ and that

$$\mathrm{Var}_{q_n}\left(\sum_{k \in K_{n,m}} k(M_k^{[\ell]} + M_k^{[r]})\right) = \sum_{k \in K_n} 2\frac{k^2 q_n^k}{(1-q_n^k)^2} = o(n^{3/2}). \tag{1}$$

Then as $n \to \infty$

$$d_{TV}\left((P_{n,m}(X^{-1}), Q_{q_n,m}(X^{-1})\right) \to 0. \tag{2}$$

*Furthermore, if (**??**) is uniform for $a_n \le m \le b_n$, then so is (**??**).*

Theorem (B.–Bringmann, Adv. Math. 2022)

Let $B := \frac{\sqrt{3}}{\pi}$. Then

$$\lim_{n\to\infty} P_n\left(\frac{\mathrm{PK} - B\sqrt{n}\log\left(2B\sqrt{n}\right)}{B\sqrt{n}} \leq v\right) = e^{-e^{-v}}.$$

Note: $n \mapsto \frac{n}{2}$ in Erdős-Lehner's Theorem plus an extra log(2).

Theorem (B.–Bringmann, Adv. Math. 2022)

Let $v_0 \geq v_1^{[j]} \geq \cdots \geq v_{t_n}^{[j]}$ for $j \in \{\ell, r\}$ and $t \leq t_n = o(n^{1/4})$. Then

$$P_n\left(\frac{\mathrm{PK} - B\sqrt{n}\log\left(2B\sqrt{n}\right)}{B\sqrt{n}} \leq v_0 \frac{L_t^{[j]} - B\sqrt{n}\log\left(2B\sqrt{n}\right)}{B\sqrt{n}} \leq v_t^{[j]},\right.$$

$$\left. 1 \leq t \leq t_n, j \in \{L, R\}\right)$$

$$\sim \int_{-\infty}^{v_0} \cdots \int_{-\infty}^{v_{t_n}^{[r]}} \frac{1}{2^{2t_n}} e^{-u_0 - \sum_{t=1}^{t_n}\left(u_t^{[\ell]} + u_t^{[r]}\right) - \frac{e^{-u_{t_n}^{[\ell]}}}{2} - \frac{e^{-u_{t_n}^{[r]}}}{2}} du_{t_n^{[r]}} \cdots du_0.$$

Remark

Away from the mean, the sequences $\mathrm{PK}, L_1^{[j]}, \ldots, L_{t_n}^{[j]}$ for $j \in \{\ell, r\}$ behave as **two Markov chains** *for for $t = o(n^{1/4})$.*

Theorem (B.–Bringmann, Adv. Math. 2022)

Let $k \leq k_n = o(n^{1/4})$. Then

$$P_n\left(\frac{kM_k^{[j]}}{A\sqrt{n}} \leq v_k^{[j]}, 1 \leq k \leq k_n, j \in \{\ell, r\}\right)$$

$$\sim \prod_{k=1}^{k_n}\left(1 - e^{-v_k^{[\ell]}}\right)\left(1 - e^{-v_k^{[r]}}\right).$$

Remark

The left and right multiplicities of $k = o(n^{1/4})$, when rescaled, behave as **independent exponential** *random variables with mean 1.*

Symmetry

Question

How is $\left(M_1^{[\ell]} - M_1^{[r]}, \ldots, M_{k_n}^{[\ell]} - M_{k_n}^{[r]}\right)$ *for* $k_n = o(n^{1/4})$ *distributed?*

$\chi_P :=$ indicator function of P

Corollary

Let $k_n = o(n^{1/4})$. *Then*

$$P_n\left(\frac{k(M_k^{[\ell]} - M_k^{[r]})}{B\sqrt{n}} \le v_k, \text{ for } k \le k_n\right) \sim \prod_{k \le k_n}\left(\left(1 - \frac{e^{-v_k}}{2}\right)\chi_{v_k > 0} + \frac{e^{v_k}}{2}\chi_{v_k \le 0}\right)$$

Answer

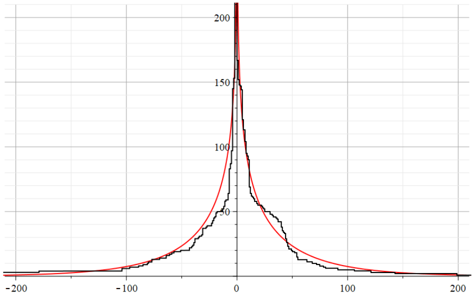
The differences $M_k^{[\ell]} - M_k^{[r]}$*, when rescaled, have independent* **Laplace distributions** *for* $k = o(n^{1/4})$. *(convolution of geometric)*

Application: sampling large unimodal sequences

1. Sample a peak according to $PK \sim \text{Gumbel}(0,1)$.
2. Generate **independent** pairs of sample multiplicities according to $M_k^{[\ell]}, M_k^{[r]} \sim \text{Geometric}(1-q_n^k)$ for $k \leq PK$.

Example

Let $n = 10,000$. I get the following u.s. with size 8393 and peak 915.



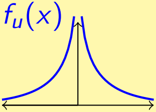
Limit shapes

Theorem (B., Int. J. Number Theory 2023)

Let $B := \frac{\sqrt{3}}{\pi}$. *Define*

$$f_u(x) := \begin{cases} -B\log\left(1 - e^{Bx}\right) & \text{if } x < 0, \\ -B\log\left(1 - e^{-Bx}\right) & \text{if } x > 0. \end{cases}$$

Then this is a limit shape for unimodal sequences under the scaling $\frac{1}{\sqrt{n}}$.



Concluding remarks

Boltzmann model useful with other product/sum-of-products generating functions and w.r.t. other *multiplicative* measures:

- partitions under multiplicative measures (Vershik, 1996)
- Rogers–Ramanujan partitions (Bogachev–Yakubovich, 2019)
- distinct parts with bounded largest part (B. 2020)
- concave compositions (Dalal–Lohss–Parry, 2021)
- strongly unimodal sequences (B. 2022)
- $\vdots$

Boltzmann model not always useful:

- Product gen. fn., but hard combinatorics.
  E.g. *plane partitions:*

  $$\sum_{n\geq 0} pp(n)q^n = \prod_{k\geq 1} \frac{1}{(1-q^k)^k}.$$

  But what does each factor count?

- Statistic not easily described in terms of M_k.
  E.g. the number of *hooks* equal to t. [Griffin–Ono–Tsai, 2022] use Method of Moments

Thanks for listening!

- W. Bridges and K. Bringmann "Statistics for unimodal sequences," *Advances in Mathematics* **401** (2022). `arXiv: 2106.02334`

- W. Bridges, "Partitions into distinct parts with bounded largest part," *Research in Number Theory* **6** (2020) `arXiv:2004.12036`

- W. Bridges, "Limit shapes for unimodal sequences," *International Journal of Number Theory* (2023), `arXiv: 2001.06878`