Analytic aspects of partitions with parts separated by parity

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Notation

The partition function is given by

$$p(n) := \# \text{ partitions of } n.$$

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1 \implies p(4) = 5.$$

Partitions in Number Theory

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Theorem (Hardy-Ramanujan, 1918)

We have that

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Theorem (Ramanujan, 1919)

For every n, we have that

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

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$$(q;q)_\infty = \sum_{n\geq 0} \left(D_{\mathrm{e}}(n) - D_{\mathrm{o}}(n)\right) q^n = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{n(3n-1)}{2}}.$$

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Theorem (Kim–Kim–Lovejoy, 2021)

Let $p_{e/o}(n)$ be the number of partitions of n with more even parts than odd parts (resp. more odd parts than even parts). Then we have

$$\frac{p_o(n)}{p_e(n)} \to 1+\sqrt{2}.$$

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A family $\mathcal{S}$ has *parts separated by parity* (PSP) if membership in $\mathcal{S}$ is partly or wholly determined by the condition above.

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$$\begin{aligned}
\sum_{n\geq 0}\mathcal{EO}(n)q^n &= \sum_{n\geq 0}\frac{q^{2n}}{(q^2;q^2)_n\,(q^{2n+1};q^2)_\infty}\\
&= \frac{1}{(q;q^2)_\infty}\sum_{n\geq 0}\frac{(q;q^2)_n}{(q^2;q^2)_n}q^{2n}\\
&= \frac{(q^3;q^2)_\infty}{(q;q^2)_\infty\,(q^2;q^2)_\infty}\\
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$$\sum_{n \geq 0} \overline{\mathcal{EO}}(n) q^n = \frac{1}{2} \left(\nu(q) + \nu(-q) \right).$$

Furthermore, we have

$$\overline{\mathcal{EO}}\,(10n+8) \equiv 0 \pmod{5},$$

and this congruence is explained by an "even-odd crank".

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- Andrews, *Partitions with Parts Separated by Parity*, 2019.
- Bringmann, Jennings-Shaffer, *A Note on Andrews' Partitions with Parts Separated by Parity*, 2019.

Notation for PSP Partitions

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We consider the following eight functions:

$$p^{\mathrm{ou}}_{\mathrm{eu}}(n),\ p^{\mathrm{od}}_{\mathrm{eu}}(n),\ p^{\mathrm{eu}}_{\mathrm{ou}}(n),\ p^{\mathrm{ed}}_{\mathrm{ou}}(n),\ p^{\mathrm{ou}}_{\mathrm{ed}}(n),\ p^{\mathrm{od}}_{\mathrm{ed}}(n),\ p^{\mathrm{eu}}_{\mathrm{od}}(n),\ p^{\mathrm{ed}}_{\mathrm{od}}(n).$$

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Observe that $\mathcal{EO}(n) = p_{\mathrm{eu}}^{\mathrm{ou}}(n)$.

Asymptotics for PSP Partitions

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$$p_{\mathrm{eu}}^{\mathrm{ou}}(n) \sim \frac{e^{\pi\sqrt{\frac{n}{3}}}}{2\pi\sqrt{n}}, \qquad p_{\mathrm{ou}}^{\mathrm{eu}}(n) \sim \frac{3^{\frac{1}{4}} e^{\pi\sqrt{\frac{n}{3}}}}{2\pi n^{\frac{1}{4}}},$$

$$p_{\mathrm{eu}}^{\mathrm{od}}(n) \sim \frac{e^{\pi\sqrt{\frac{n}{3}}}}{4\sqrt{2}\cdot 3^{\frac{1}{4}} n^{\frac{3}{4}}}, \qquad p_{\mathrm{od}}^{\mathrm{eu}}(n) \sim \frac{e^{\pi\sqrt{\frac{n}{3}}}}{2\sqrt{3}n},$$

$$p_{\mathrm{ed}}^{\mathrm{ou}}(n) \sim \frac{e^{\pi\sqrt{\frac{n}{3}}}}{4\cdot 3^{\frac{1}{4}} n^{\frac{3}{4}}}, \qquad p_{\mathrm{ou}}^{\mathrm{ed}}(n) \sim \frac{e^{\pi\sqrt{\frac{n}{3}}}}{4\sqrt{2}\sqrt{n}},$$

$$p_{\mathrm{ed}}^{\mathrm{od}}(n) \sim \frac{3^{\frac{1}{4}}\left(\sqrt{2}-1\right) e^{\pi\sqrt{\frac{n}{6}}}}{2^{\frac{3}{4}}\pi n^{\frac{1}{4}}}, \qquad p_{\mathrm{od}}^{\mathrm{ed}}(n) \sim \frac{3^{\frac{1}{4}}\left(\sqrt{2}-1\right) e^{\pi\sqrt{\frac{n}{6}}}}{2^{\frac{1}{4}}\pi n^{\frac{1}{4}}}.$$

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We define the generating functions $F_{\mathrm{yz}}^{\mathrm{wx}}(q) := \sum_{n \geq 0} p_{\mathrm{yz}}^{\mathrm{wx}}(n) q^n$.

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Example

We have the following constructions:

$$F_{\mathrm{eu}}^{\mathrm{od}}(q) = \sum_{n\geq 0} \frac{\left(-q^{2n+1};q^2\right)_\infty}{\left(q^2;q^2\right)_n} q^{2n};$$

$$F_{\mathrm{od}}^{\mathrm{eu}}(q) = \sum_{n\geq 0} \frac{\left(-q;q^2\right)_n}{\left(q^{2n+2};q^2\right)_\infty} q^{2n+1} + \frac{1}{\left(q^2;q^2\right)_\infty}.$$

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All eight generating functions can be constructed using q-hypergeometric series in this very classical manner.

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$$F_{\mathrm{ed}}^{\mathrm{od}}(q) = \frac{\left(-q;q^2\right)_\infty}{1-q} - \frac{q\left(-q^2;q^2\right)_\infty}{1-q},$$

$$F_{\mathrm{ou}}^{\mathrm{eu}}(q) = \frac{1}{1-q}\left(\frac{1}{\left(q;q^2\right)_\infty} - \frac{q}{\left(q^2;q^2\right)_\infty}\right),$$

$$F_{\mathrm{od}}^{\mathrm{ed}}(q) = \frac{(1+q)\left(-q^2;q^2\right)_\infty}{1-q} - \frac{q\left(-q;q^2\right)_\infty}{1-q}.$$

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$$F_{\mathrm{ed}}^{\mathrm{ou}}(-q) = \frac{1}{2(-q;q^2)_\infty} \left((-q;q)_\infty + 1 - \sum_{n \geq 0} (1-q^n)\, q^{\frac{n(3n-1)}{2}} \right).$$

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Proposition (Mock PSP's)

The following generating functions hold:

$$F_{\mathrm{ou}}^{\mathrm{ed}}(-q) = \frac{(-q^2;q^2)_\infty}{2} \left(2 - f(q) + \frac{1}{(-q;q)_\infty} \right).$$

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Remark

We will return to $F_{\mathrm{od}}^{\mathrm{eu}}(q)$ later...

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Let $B(q) = \sum_{n \geq 0} b(n) q^n$ be a power series whose radius of convergence is at least one and assume that $b(n)$ are non-negative and weakly increasing.

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$$B\left(e^{-t}\right) \sim \lambda t^{\beta} e^{\frac{\gamma}{t}} \quad \text{as } t \to 0^+, \qquad B\left(e^{-z}\right) \ll |z|^{\beta} e^{\frac{\gamma}{|z|}} \quad \text{as } z \to 0,$$

with the latter condition holding in each region of the form $|y| \leq \Delta x$ *for* $\Delta > 0$ *and* $z = x + iy$ *with* $x, y \in \mathbb{R}$, $x > 0$.

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$$b(n) \sim \frac{\lambda \gamma^{\frac{\beta}{2} + \frac{1}{4}}}{2\sqrt{\pi} n^{\frac{\beta}{2} + \frac{3}{4}}} e^{2\sqrt{\gamma n}} \qquad \text{as } n \to \infty.$$

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Remark

For PSP's, the parity separation condition is convenient for proving "suitable" increasing properties.

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The Dedekind η-function is defined for $\tau \in \mathbb{C}$ satisfying $\mathrm{Im}(\tau) > 0$ by

$$\eta(\tau) = q^{\frac{1}{24}}(q;q)_\infty, \quad \Theta(q) := \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}} \quad (q = e^{2\pi i\tau}).$$

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Lemma

Let $q = e^{-z}$. Then as $z \to 0$ in regions $|y| \leq \Delta x$ for $\Delta > 0$ and $z = x + iy$, we have the asymptotic behaviors

$$(q;q)_\infty \sim \sqrt{\frac{2\pi}{z}} e^{-\frac{\pi^2}{6z}}, \quad \Theta(q) \sim \sqrt{\frac{2\pi}{z}}.$$

Asymptotics for infinite series

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Proposition (Euler–Maclaurin summation)

Let g be a holomorphic function in a domain containing those $z = x + iy$ satisfying $|y| \leq \Delta x$, $x \geq 0$. Also suppose that g, as well as all of its derivatives, are of sufficient decay.

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$$\sum_{m \geq 0} g((m+a)z) = \frac{1}{z} \int_0^\infty g(w)dw - \sum_{n=0}^{N-1} \frac{B_{n+1}(a) g^{(n)}(0)}{(n+1)!} z^n + O_N\left(z^N\right),$$

as $z \to 0$ uniformly in this region. Here $B_n(x)$ denotes the n-th Bernoulli polynomial.

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Remark

Can be used to study partial ϑ-functions after completing the square in the exponent.

The Case of $F_{\mathrm{od}}^{\mathrm{eu}}$

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  - Modular forms (Hardy–Ramanujan, Rademacher)
  - Mock modular forms (Zwegers, Bringmann–Ono)
  - Partial/false ϑ-functions (Bringmann–Nazaroglu, 2019)
- The function $F^{\mathrm{eu}}_{\mathrm{od}}(q)$ involves *mock Maass forms*, which have not previously been studied in this way.

Ramanujan's σ-function

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We define Ramanujan's σ-function by

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We have the generating function

$$\sigma(q) = \sum_{\substack{n \geq 0 \\ |j| \leq n}} (-1)^{n+j} \left(1 - q^{2n+1}\right) q^{\frac{n(3n+1)}{2} - j^2}.$$

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Remark

Observe that this is a false indefinite ϑ-function.

Connections to PSP's

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Theorem

We have the generating function identity

$$F_{\mathrm{od}}^{\mathrm{eu}}(-q) = -\frac{1}{(q^2;q^2)_\infty}\left(\sum_{j\geq 1}\sum_{n\geq j}(-1)^{n+j}\left(1-q^{2n+1}\right)q^{\frac{n(3n+1)}{2}-j^2}-1\right)$$

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Question

What is the modular structure of $\sigma(q)$?

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Theorem (Cohen, 1988)

The nonholomorphic series ($q = e^{-z} = e^{-x-iy}$)

$$\varphi_0(q) := y^{1/2} \sum_{n\in\mathbb{Z}\setminus\{0\}} T(n) K_0\left(\frac{2\pi|n|y}{24}\right) e^{\frac{2\pi inx}{24}}$$

is an eigenvalue of the hyperbolic Laplacian $\Delta := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ with eigenvalue $\frac{1}{4}$ and transforms as a modular form with multiplier for $\Gamma_0(2)$.

Mock Maass Forms: Notation

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- Since Q has signature $(1, 1)$, we can choose P so that

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- Let

$$c(t) = P^{-1} \begin{pmatrix} \exp(t) \\ -\exp(-t) \end{pmatrix}, \quad c^{\perp}(t) = P^{-1} \begin{pmatrix} \exp(t) \\ \exp(-t) \end{pmatrix}.$$

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- For fixed c_0, let $C_Q := \{c \in \mathbb{R}^2 : Q(c) = -1, B(c, c_0) < 0\}$; $c(t)$ parameterizes C_Q, $c^{\perp}(t)$ its complement, and we choose t_1, t_2 and set $c(t_i) = c_i$, $c^{\perp}(t_i) = c_i^{\perp}$.

False indefinite quadratic forms

False indefinite quadratic forms

- Using the previous notation, we consider the *false indefinite ϑ-functions*

$$\vartheta_\mu(\tau) := \frac{1}{2} \sum_{\substack{\mathbf{n} \in \mathbb{Z}^2+\mu \\ \mathbf{n} \neq \mathbf{0}}} \left(1 - \operatorname{sgn}\left(B(\mathbf{n}, \mathbf{c}_1)\right) \operatorname{sgn}\left(B(\mathbf{n}, \mathbf{c}_2)\right)\right) q^{Q(\mathbf{n})}$$

$$- \frac{t_2 - t_1}{\pi} \delta_{\mu \in \mathbb{Z}^2}$$

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- In our PSP study, we will make use of the example associated with $A = \begin{pmatrix} 24 & 0 \\ 0 & 4 \end{pmatrix}$:

$$f_\mu(\tau) := \frac{1}{2} \sum_{\mathbf{n}\in\mathbb{Z}^2+\mu} \left(1 + \operatorname{sgn}\left(2n_1 + n_2\right)\operatorname{sgn}\left(2n_1 - n_2\right)\right) q^{12n_1^2 - 2n_2^2} - \frac{\operatorname{arccosh}(5)}{\pi}\delta_{\mu\in\mathbb{Z}^2}.$$

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For $\mu \in A^{-1}\mathbb{Z}^2/\mathbb{Z}^2$, we define the *mock Maass theta functions* associated to $\vartheta_\mu(\tau)$ by (with $\tau = \tau_1 + i\tau_2$) by

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$$\Theta_\mu(\tau) = \frac{\sqrt{\tau_2}}{2} \sum_{\substack{\mathbf{n}\in\mathbb{Z}^2+\mu \\ \mathbf{n}\neq\mathbf{0}}} \left(1 - \operatorname{sgn}\left(B(\mathbf{n},\mathbf{c}_1)\right)\operatorname{sgn}\left(B(\mathbf{n},\mathbf{c}_2)\right)\right) K_0\left(2\pi Q(\mathbf{n})\tau_2\right) e^{2\pi i Q(\mathbf{n})\tau_1}$$
$$+ \frac{\sqrt{\tau_2}}{2} \sum_{\substack{\mathbf{n}\in\mathbb{Z}^2+\mu \\ \mathbf{n}\neq\mathbf{0}}} \left(1 - \operatorname{sgn}\left(B(\mathbf{n},\mathbf{c}_1^\perp)\right)\operatorname{sgn}\left(B(\mathbf{n},\mathbf{c}_2^\perp)\right)\right) K_0\left(-2\pi Q(\mathbf{n})\tau_2\right) e^{2\pi i Q(\mathbf{n})\tau_1}$$
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$$+ (t_2 - t_1)\sqrt{\tau_2}\,\delta_{\mu\in\mathbb{Z}^2}.$$

- We note that Θ_μ is an eigenvalue of the hyperbolic Laplacian.
- We will use $F_\mu(\tau)$ to denote the mock Maass theta function associated to $f_\mu(\tau)$.

Modular Completions

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Definition

Define the *modular completion* of $\Theta_\mu(\tau)$ by

$$\widehat{\Theta}_\mu(\tau) := \sqrt{\tau_2} \sum_{\mathbf{n} \in \mathbb{Z}^2 + \mu} q^{Q(\mathbf{n})} \int_{t_1}^{t_2} e^{-\pi B(\mathbf{n}, \mathbf{c}(t))^2 \tau_2} dt.$$

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Theorem (Zwegers, 2012)

For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, *we have*

$$\widehat{\Theta}_\mu(M\tau) = \sum_{\nu\in A^{-1}\mathbb{Z}^2/\mathbb{Z}^2} \psi_M(\mu,\nu)\,\widehat{\Theta}_\nu(\tau).$$

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for a certain multiplier system ψ. *Furthermore, the difference* $\widehat{\Theta}_\mu - \Theta_\mu$ *is explicit, and in many cases vanishes, in which case the mock Maass form is a Maass form.*

Connections to PSP's

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- Using Andrews–Dyson–Hickerson, it is known that

$$F_{\mathrm{od}}^{\mathrm{eu}}(q) = \frac{1}{(q^2;q^2)_\infty} + \frac{(-q;-q)_\infty}{2\,(q^2;q^2)_\infty} - \frac{\sigma(-q)}{2\,(q^2;q^2)_\infty}.$$

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- Letting $p(n)$ and $\mathrm{sc}(n)$ count partitions and self-conjugate partitions, define

$$\alpha_0(n) = 2p_{\mathrm{od}}^{\mathrm{eu}}(2n) - 2p(n) - \mathrm{sc}(2n),$$
$$\alpha_1(n) = 2p_{\mathrm{od}}^{\mathrm{eu}}(2n+1) - \mathrm{sc}(2n+1).$$

Then

$$\sum_{n\geq 0} \alpha_0(n) q^{2n} + \sum_{n\geq 0} \alpha_1(n) q^{2n+1} = -\frac{\sigma(-q)}{(q^2;q^2)_\infty}$$

Connection to PSP's

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- For $u_0(\tau) = -q^{\frac{1}{48}} \frac{\sigma(q)+\sigma(-q)}{2}$ and $u_1(\tau) = q^{\frac{1}{48}} \frac{\sigma(q)-\sigma(-q)}{2}$,

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- Using the Andrews–Dyson–Hickerson, we relate u_0, u_1 to false indefinite ϑ-functions by

$$\begin{aligned} u_0 &= -f_{\left(\frac{1}{24},0\right)} + f_{\left(\frac{7}{24},0\right)} + f_{\left(\frac{13}{24},\frac{1}{2}\right)} - f_{\left(\frac{19}{24},\frac{1}{2}\right)} \\ u_1 &= -f_{\left(\frac{1}{24},\frac{1}{2}\right)} + f_{\left(\frac{7}{24},\frac{1}{2}\right)} + f_{\left(\frac{13}{24},0\right)} - f_{\left(\frac{19}{24},0\right)}. \end{aligned}$$

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$$u_1 = -f_{\left(\frac{1}{24},\frac{1}{2}\right)} + f_{\left(\frac{7}{24},\frac{1}{2}\right)} + f_{\left(\frac{13}{24},0\right)} - f_{\left(\frac{19}{24},0\right)}.$$
- Using $f_\mu = f_{-\mu}$, we can naturally write for $0 \leq j \leq 2$:
$$u_j = \frac{1}{2}\left(\sum_{\mu\in\mathcal{S}_j^+} f_\mu - \sum_{\mu\in\mathcal{S}_j^-} f_\mu\right).$$

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Lemma (Bringmann–C–Nazaroglu)

We define for $0 \leq j \leq 2$

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For each j*, we have* $U_j = \widehat{U}_j$.

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For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$*, we have*

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for a certain multiplier system $\Psi_M(j,k)$*.*

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Remark

Follows from the mock Maass form theory.

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Remark

It is crucial to understand the size of $u_k(\tau) + \mathcal{E}_{k-\frac{d}{c}}(\tau)$.

Mordell-type Representation

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- We use the Fourier expansions

$$u_j(\tau) = \sum_{\substack{n \in \mathbb{Z}+\alpha_j \\ n>0}} d_j(n) q^n,$$

$$U_j(\tau) = \sqrt{\tau_2} \sum_{n \in \mathbb{Z}+\alpha_j} d_j(n) K_0\left(2\pi |n| \tau_2\right) e^{2\pi i n \tau_1}.$$

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$$\mathcal{E}_{k,-\frac{d}{c}}(\tau) = -\frac{1}{\pi} \int_0^\infty \sum_{n \in \mathbb{Z}+\alpha_k} d_k(n) e^{-\frac{2\pi i d n}{c}} \frac{t K_0\left(2\pi |n| t\right)}{\sqrt{t^2 + \left(\tau + \frac{d}{c}\right)^2}} \cdot \left(2\pi n + \frac{i\left(\tau + \frac{d}{c}\right)}{t^2 + \left(\tau + \frac{d}{c}\right)^2}\right) dt.$$

Mordell-type Representation

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- Problem: Absolute convergence not clear for sum-integral swap

Mordell-type Representation

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Lemma (Bringmann–C–Nazaroglu)

We have

$$\mathcal{E}_{k,-\frac{d}{c}}(\tau) = -\frac{1}{2\pi^2\left(\tau+\frac{d}{c}\right)} \lim_{\delta\to 0^+} \sum_{n\in\mathbb{Z}+\alpha_k} \frac{d_k(n) e^{-\frac{2\pi i d n}{c}}}{n} \mathcal{K}\left(2\pi|n|\delta\right)$$
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where $\mathcal{K}(x) := xK_1(x)$ *and*

$$\mathcal{K}_{\tau,\frac{d}{c}}(n) = \operatorname{sgn}(n) f\left(2\pi|n|\left(\tau+\frac{d}{c}\right)\right) + ig\left(2\pi|n|\left(\tau+\frac{d}{c}\right)\right) - \frac{1}{2\pi n\left(\tau+\frac{d}{c}\right)}$$

for

$$f(w) := i\mathrm{PV}\int_0^\infty \frac{e^{iwt}}{t^2-1}dt + \frac{\pi}{2}e^{iw}, \quad g(w) := \mathrm{PV}\int_0^\infty \frac{te^{iwt}}{t^2-1}dt - \frac{\pi i}{2}e^{iw}.$$

Mordell-type Representation

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Proposition (Bringmann–C–Nazaroglu)

Define the function

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Mordell-type Representation

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Then we have

$$u_k(\tau) + \mathcal{E}_{k,-\frac{d}{c}}(\tau) = \mathcal{I}_{k,-\frac{d}{c}}(\tau).$$

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Question

What is the “principal part” of $\mathcal{I}_{k,-\frac{d}{c}}(\tau)$?

Finding the Principal Part

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- We fix the notation

$$\frac{u_j(\tau)}{\eta(\tau)} = \sum_{n=0}^{\infty} \alpha_j(n) q^{n+\Delta_j}, \ \Delta_0 := -\frac{1}{48}, \ \Delta_1 := \frac{23}{48}, \ \Delta_2 := \frac{11}{12}.$$

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- By Cauchy's theorem, we have

$$\alpha_j(n) = \int_i^{i+1} \frac{u_j(\tau)}{\eta(\tau)} e^{-2\pi i \left(n+\Delta_j\right)\tau} d\tau,$$

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- Goal: Estimate this integral using Rademacher's techniques.

Circle Method: Rademacher's Path

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- Using Rademacher's path of integration (i.e. using Farey arcs of order N and Ford circles) we have

$$\alpha_j(n) = i \sum_{k=1}^{N} k^{-2} \sum_{\substack{0 \leq h < k \\ \gcd(h,k)=1}} \int_{Z_1}^{Z_2} \frac{u_j\left(\frac{h}{k} + \frac{iZ}{k^2}\right)}{\eta\left(\frac{h}{k} + \frac{iZ}{k^2}\right)} e^{-2\pi i(n+\Delta_j)\left(\frac{h}{k}+\frac{iZ}{k^2}\right)} dZ,$$

where $\tau = \frac{h}{k} + \frac{iZ}{k^2}$ and Z_1, Z_2 are certain points on the circle of radius $\frac{1}{2}$ and center $\frac{1}{2}$.

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- Using previously derived modular transformations and $\tau = \frac{h'}{k} + \frac{i}{Z}$ we will apply the calculation

$$u_j\left(\frac{h}{k}+\frac{iZ}{k^2}\right) = \frac{ik}{Z}\sum_{\ell=0}^{2} \Psi_{M_{h,k}}(j,\ell)\mathcal{I}_{\ell,\frac{h'}{k}}\left(\frac{h'}{k}+\frac{i}{Z}\right).$$

Circle Method: Principal Parts

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- Using the modular transformation for the eta function,

$$\alpha_j(n) = \sum_{\ell=0}^{2} \sum_{k=1}^{N} k^{-\frac{3}{2}} \sum_{\substack{0 \le h < k \\ \gcd(h,k)=1}} \frac{e^{\frac{3\pi i}{4}} \Psi_{M_{h,k}}(j,\ell)}{\nu_\eta(M_{h,k})}$$

$$\cdot \int_{Z_1}^{Z_2} Z^{-\frac{1}{2}} \frac{\mathcal{I}_{\ell,\frac{h'}{k}}\left(\frac{h'}{k} + \frac{i}{Z}\right)}{\eta\left(\frac{h'}{k} + \frac{i}{Z}\right)} e^{-2\pi i (n+\Delta_j)\left(\frac{h}{k} + \frac{iZ}{k^2}\right)} dZ.$$

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- We now split off the principal parts using Now we split off the principal part contributions by writing

$$\frac{\mathcal{I}_{\ell,\frac{h'}{k}}\left(\frac{h'}{k} + \frac{i}{Z}\right)}{\eta\left(\frac{h'}{k} + \frac{i}{Z}\right)} = e^{-\frac{\pi i h'}{12k}} \mathcal{I}^*_{\ell,\frac{h'}{k},\frac{1}{24}}\left(\frac{h'}{k} + \frac{i}{Z}\right) + e^{-\frac{\pi i h'}{12k}} \mathcal{I}^e_{\ell,\frac{h'}{k},\frac{1}{24}}\left(\frac{h'}{k} + \frac{i}{Z}\right)$$

$$+ \mathcal{I}_{\ell,\frac{h'}{k}}\left(\frac{h'}{k} + \frac{i}{Z}\right)\left(\frac{1}{\eta\left(\frac{h'}{k} + \frac{i}{Z}\right)} - e^{-\frac{\pi i}{12}\left(\frac{h'}{k}+\frac{i}{Z}\right)}\right).$$

Circle Method: Error Estimation

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- After estimating the error terms and setting $N = \lfloor \sqrt{n} \rfloor$, we obtain

$$\alpha_j(n) = \sum_{\ell=0}^{2} \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} k^{-\frac{3}{2}} \sum_{\substack{0 \leq h < k \\ \gcd(h,k)=1}} \frac{e^{\frac{3\pi i}{4}} \Psi_{M_{h,k}}(j,\ell)}{\nu_\eta(M_{h,k})} e^{-\frac{\pi i h'}{12k}}$$
$$\times \int_{Z_1}^{Z_2} Z^{-\frac{1}{2}} \mathcal{I}^*_{\ell, \frac{h'}{k}, \frac{1}{24}} \left(\frac{h'}{k} + \frac{i}{Z} \right) e^{-2\pi i (n+\Delta_j)\left(\frac{h}{k} + \frac{iZ}{k^2}\right)} dZ + O\left(n^{\frac{3}{4}}\right).$$

Final Theorem

Final Theorem

Theorem (Bringmann–C–Nazaroglu)

We have

$$\alpha_j(n) = 2(n+\Delta_j)^{-\frac{1}{4}} \sum_{\ell=0}^{2} \sum_{k=1}^{\lfloor\sqrt{n}\rfloor} \frac{1}{k} \sum_{\substack{0\le h<k \\ \gcd(h,k)=1}} \psi_{h,k}(j,\ell)$$

$$\times \mathrm{PV}\int_0^{\frac{1}{24}} \Phi_{\ell,\frac{h'}{k}}(t)\left(\frac{1}{24}-t\right)^{\frac{1}{4}} I_{\frac{1}{2}}\left(\frac{4\pi}{k}\sqrt{(n+\Delta_j)\left(\frac{1}{24}-t\right)}\right) dt + O\left(n^{\frac{3}{4}}\right).$$

where

$$\Phi_{\ell,\frac{h'}{k}}(t) := \sum_{n\in\mathbb{Z}+\alpha_\ell}^{*} \frac{d_\ell(n)e^{\frac{2\pi i h' n}{k}}}{t-n}$$

Open Questions

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- Combinatorial explanation for inequalities between PSP's:

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- Modifications with congruence properties similar to $\overline{\mathcal{EO}}(n)$?

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- Modifications with congruence properties similar to $\overline{\mathcal{EO}}(n)$?

- Connections between hypergeometric representations and Jacobi properties?

End of Talk

Questions?