# Local-to-global principles for Sturm bounds

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## Notation

*The partition function is given by*

$$p(n) := \# \text{ partitions of } n.$$

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1 \implies p(4) = 5.$$

# Partitions in Number Theory

## Theorem (Hardy–Ramanujan, 1918)

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## Theorem (Ramanujan, 1919)

*For every n, we have that*

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7},$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

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- and so on...

# Radu's algorithm (Bird's eye view)

**Theorem**

*Let* $\prod_{\delta|M}\prod_{n=1}^{\infty}\left(1-q^{\delta n}\right)^{r_\delta}=\sum_{n\geq 0}a(n)q^n$. *Then there is an explicit, finite time algorithm where, given positive integers* $m\geq 1$, $0\leq t<m$, *and a prime* $p$, *which outputs whether* $a(mn+t)\equiv 0 \pmod{p}$ *for all* $n\geq 0$.

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- This is essentially a modular form of a known weight and known congruence subgroup.
- If it has poles at $i\infty$, multiply by a "suitably chosen" modular form so that the result is now a holomorphic modular form of positive weight.
- The *Sturm bound* says that congruences for these can be checked in explicit finite time.

□

# The Sturm Bound

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For a $q$-series $\sum_{n \gg 0} a(n)q^n$ over any ring, define

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*Let $k \geq 1$ be an integer and let $\Gamma$ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ with finite index $d$. Let $f$ be a holomorphic modular form for $\Gamma$ of weight $k$ with integer coefficients. The following are true:*

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2. *If $f(q) \in (\mathbb{Z}/m\mathbb{Z})[[q]]$ for $m \geq 1$ and $\operatorname{ord}_\infty(f) > \frac{dk}{12}$, then $f \equiv 0 \pmod{m}$.*

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- and many more (mathematical physics, $L$-functions, Iwasawa theory, ...)

# Proof of Sturm's theorem

- If $f(\tau)$ is a holomorphic modular form on $\mathrm{SL}_2(\mathbb{Z})$ of weight $k$, then the *valence formula* says that

$$\sum_{\tau} \frac{1}{e_\tau} \mathrm{ord}_\tau(f) = \frac{k}{12},$$

where $e_\tau$ is the ramification of $f$ at $\tau$ (2 and $i$, 3 at a cube root of unity, 1 elsewhere).

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- A zero of order $> \frac{k}{12}$ at zero contradicts this formula, since $f$ is holomorphic (no poles to balance any extra zeros).
- For a congruence subgroup $\Gamma$, the *traceform*

$$g := \prod_{\gamma \in \mathrm{SL}_2(\mathbb{Z})/\Gamma} f|_k \gamma$$

  is a modular form on $\mathrm{SL}_2(\mathbb{Z})$ of weight $dk$ and $\mathrm{ord}_\infty(g) = d\mathrm{ord}_\infty(f)$.

# Where does Sturm come from?

**Theorem (Riemann–Roch Theorem)**

*Let $X$ be a compact Riemann surface of genus $g$, and let $D$ be a divisor on $X$. Then we have*

$$l(D) - l(K - D) = \deg(D) + 1 - g,$$

*where:*

- $l(D)$ *is the dimension of the space of meromorphic functions* $f$ *such that* $(f) + D \geq 0$.
- $K$ *is the canonical divisor.*
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## Corollary

*Valence formulas are Riemann–Roch for the modular curves $\Gamma\backslash\mathbb{H}$.*

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**Local-to-global principles** "Mod $p$ for all $p \implies \mathbb{Z}$"

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- Identify **local relationship** with modular forms (or some other objects) that obey a (local) Sturm-type bound.
- Show coefficients of the new non-geometric forms have explicitly limited growth rates.
- **[Chinese Remainder Theorem + Linear Algebra]** glue together the **local Sturm bounds** into a **global Sturm bound**.

## Definition

Let $k \geq 1$ be an integer.

- The **Eisenstein series** of weight $2k$ is

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- A **quasimodular form** of weight $2k$ is any analytic function $f(z)$ of the form

$$f = \sum_{j=0}^{k} E_2^j f_j,$$

where $f_j \in M_{2k-2j}$. The space of such forms of weight $2k$ is $\widetilde{M}_{2k}$.

# A New Sturm Theorem

**Theorem (C, 2026)**

*Let* $k \geq 1$ *be an integer, and let* $f \in \mathbb{Z}[[q]] \cap M_k$ *be a quasimodular form. Then if* $\mathrm{ord}_\infty(f) > \left\lceil \frac{k^4 \log(k)}{2} \right\rceil$, *then* $f = 0$.

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- Therefore, *there is not underlying geometry to use*.

# Local relationship with modular forms

## Proposition

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## Proof.

By Fermat's little theorem and the Kummer congruences for Bernoulli numbers,

$$E_{p+1} \equiv E_2 \pmod{p} \quad \text{and} \quad E_{p-1} \equiv 1 \pmod{p}.$$

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$$E_{p+1} \equiv E_2 \pmod{p} \quad \text{and} \quad E_{p-1} \equiv 1 \pmod{p}.$$

Thus, if $f = \sum_{j=0}^{k} f_j E_2^j$ for some modular forms $f_j \in M_{2k-2j}$, we have

$$f \equiv \sum_{j=0}^{k} f_j E_{p+1}^j E_{p-1}^{2k-2j} \pmod{p},$$

which is modular of weight exactly $2kp$. □

# Growth of coefficients

## Lemma

*Let $a_n, b_n$ be two sequences with $a_0 = b_0 = 1$, $|a_n| \leq An^{\alpha}(1 + \log(n))^{\gamma}$, and $|b_n| \leq Bn^{\beta}\,(1 + \log(n))^{\delta}$ for $1 \leq n \leq N$ and $\alpha, \beta, \gamma > 0$ integers. Then if $c_n$ is the convolution of $a_n$ and $b_n$, we have*

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## Remark

*Uniform bound valid for each element of a basis.*

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- Suppose that the first $m$ coefficients ($m$ unknown) were enough to determine an element of $\widetilde{M}_{2k} \cap \mathbb{Z}[[q]]$ uniquely. Let $d = \dim \widetilde{M}_{2k}$.

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- **Restart this argument with new $m$ if smaller than old $m$!**

# Example of Algorithm

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# Proof of Key Lemma

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- Therefore, $\det(N) \equiv 0 \pmod{P}$.

# Proof of Key Lemma

- This is true if and only if $\det(N) \equiv 0 \pmod{p_j}$ for each $p_1, \ldots, p_\ell$ for suitably large chosen $p_\ell$, i.e. so that with $m < k^2 \, \mathcal{E}_k(x) \, \mathcal{E}_k(x-1)$,

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## Proof.

Apply same algorithm two more times:

$$k^{k+3} \longrightarrow O(k^5 \log(k)) \longrightarrow O(k^4 \log(k)) \longrightarrow \frac{k^4 \log(k)}{2} + 1.$$

Any further attempts at reduction only yield marginal improvements of the $O$-constant. □

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  - This is extremely far from being true if the columns are coefficients of modular forms.
  - **Thus, improvements on the Hadamard bounds in these special cases are likely to yield better Sturm bounds** (not exactly $\dim \widetilde{M}_{2k}$, but $O(k^2 \log(k))$ could be done)
- Could be done by a study of **Rankin-Selberg convolutions**, **Perron formulas**, to study "angle between columns".

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- But, the general idea of the argument would still work if extra multiplication is done.
- Stronger bounds than this will follow if one starts the argument from the beginning with a level $N$-specific basis (area for future research)

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- How do we classify and remove such relations?

# Galois representations and modular form modulo $p$

## Proposition (Serre, Swinnerton–Dyer)

*Let $k \geq 0$ be an integer, and let $M_{2k}$ and $M_{2k}^p$ denote the spaces of modular forms of weight $2k$ over $\mathbb{C}$ and $\mathbb{F}_p$, respectively. Let $M$ and $M^p$ denote the full algebras of all modular forms over $\mathbb{C}$ and $\mathbb{F}_p$, respectively. Then the following are true:*

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3. *Let $A(Q,R)$ be the isobaric weight $p-1$ polynomial such that $A(E_4, E_6) = E_{p-1}$, and let $\overline{A}$ denote the reduction of this polynomial modulo p. Then there is an isomorphism*

$$\mathbb{F}_p[Q,R]/\langle \overline{A}(Q,R) \rangle \xrightarrow{\sim} M^p.$$

# Implications for Sturm bounds

## Proposition

*Let $k \geq 10$ be an integer. Then the space $\widetilde{M}_{2k}$ has a $\mathbb{Z}$-basis with the property that for each $f$ in the basis, its Fourier coefficients $a_f(n)$ satisfy the bound*

$$|a_f(n)| \leq \left(\frac{k+6}{6}\right)^{\frac{k^2}{3}} \cdot 109 \cdot 2^{2k-1} n^{2k-1} (1 + \log(n))^k .$$

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  - "Fixing example": $\{E_2^2, E_4\} \mapsto \{\frac{E_2^2 - E_4}{12}, E_4\}$

□

# Implications for Sturm bounds

**Theorem (C, 2026)**

*Let $k \geq 1$ be an integer, and let $f \in R[[q]] \cap M_k$ be a quasimodular form with coefficients in the ring $R = \mathbb{Z}$ or $\mathbb{Z}/m\mathbb{Z}$ for $m \geq 2$ an integer. Then if $\mathrm{ord}_\infty(f) > \left\lceil \frac{k^5 \log(k)}{12} \right\rceil$, then $f = 0$.*

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- The result is uniform in the ring, unlike our first result, but has the downside of picking up an extra $O(k)$-factor in the bound.
- Numerical basis calculations suggest that the coefficients do become quite large, in which case this worsened bound might be structural and not a sign of a weak bounding process

# Summary

## Theorem (C, 2026)

*Let $k \geq 1$ be an integer, and let $f \in \mathbb{Z}[[q]] \cap M_k$ be a quasimodular form.*

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- A study of Rankin–Selberg convolutions and $L$-functions may lead to improvements in order of magnitude to the bounds.
- Lack of uniformity appears more structural, but may be repaired with improved methods for producing $\mathbb{Z}$-basis.
- Generalization to congruence subgroups $\Gamma$ will follow from a suitable analysis of a basis for $M_k(\Gamma) \cap \mathbb{Z}[[q]]$.

# End of Talk

**Questions?**