Seaweed Algebras and Partitions

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Partitions

Definition

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$$\lambda = (\lambda_1, \ldots, \lambda_k).$$

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Example

We have $p(4) = 5$ since
$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

A Theme From Euler

Fact (Euler)

- *We have*

$$\sum_{\lambda} q^{|\lambda|} = \sum_{n\geq 0} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

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- *For $\ell(\lambda) := \#$ parts of λ,*

$$\sum_{\lambda} (-1)^{\ell(\lambda)} q^{|\lambda|} = \prod_{n=1}^{\infty} \frac{1}{1+q^n}.$$

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Let $\mathcal{D} = \{\lambda : \text{distinct parts}\}$.

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Euler's identities are

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$$\sum_{\lambda\in\mathcal{D}} q^{|\lambda|} = (-q;q)_\infty \qquad \sum_{\lambda\in\mathcal{D}} (-1)^{\ell(\lambda)} q^{|\lambda|} = (q;q)_\infty$$

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We have the q-series expansions

$$(-q;q)_\infty^{-1} = \sum_{\lambda} (-1)^{\ell(\lambda)} q^{|\lambda|} = 1 - q - q^3 + q^4 - q^5 + q^6 - q^7 + 2q^8 - 2q^9 + \dots$$

and

$$(q;q)_\infty = \sum_{\lambda \in \mathcal{D}} (-1)^{\ell(\lambda)} q^{|\lambda|} = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \dots$$

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Remark

The signs of these coefficients are eventually periodic.

An Example of Coll–Mayers–Mayers

Definition

For $(a, b; q)_\infty := (a; q)_\infty (b; q)_\infty$, define

$$G(q) := \left(q, -q^3; q^4\right)_\infty^{-1} = \prod_{n=0}^{\infty} \frac{1}{1 + (-1)^{n+1} q^{2n+1}}.$$

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- *Expanding the product as a sum,*

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$$G(q) = 1 + q + q^2 + q^5 + 2q^6 + q^7 + 2q^{10} + 2q^{11} + q^{12} + q^{14} + \dots$$

- *For $\mathcal{OD} = \{\lambda : \text{odd distinct parts}\}$,*

$$G(q)^{-1} = (q, -q^3; q^4)_\infty = \sum_{\lambda \in \mathcal{OD}} (-1)^{\#\{\lambda_i \equiv 1 \bmod 4\}} q^{|\lambda|}.$$

A Natural Question

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Does $G(q)$ directly count anything?

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Conjecture (Coll, Mayers, Mayers (2018))

Yes, $G(q)$ counts a parity bias arising from certain Lie algebras.

Lie Algebras

Definition

A *Lie algebra* is a vector space $\mathfrak{g}$ along with a bilinear bracket $[\cdot, \cdot]$ satisfying $[X, Y] = -[Y, X]$ and the Jacobi identity

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0.$$

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Examples

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- Vector subspaces of $\mathfrak{gl}(n)$ closed under $[\cdot, \cdot]$.
- For example, $\mathfrak{sl}(n) := \{X \in \mathfrak{gl}(n) : \mathrm{tr}(X) = 0\}$

Lie Algebras From Partitions

Example (Partitions of 8)

Let $\lambda = (3, 3, 2)$ and $\mu = (4, 3, 1)$. We construct an 8×8 matrix X:

$$X = \begin{pmatrix} * & & & & & & & \\ & * & & & & & & \\ & & * & & & & & \\ & & & * & & & & \\ & & & & * & & & \\ & & & & & * & & \\ & & & & & & * & \\ & & & & & & & * \end{pmatrix}$$

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Definition (Dergachev, Kirillov)

Subsets of $\mathfrak{gl}(n)$ formed this way from $\lambda, \mu \vdash n$ are called *seaweed algebras*.

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A *meander* is an undirected graph G on n vertices whose connected components are all either paths or cycles.

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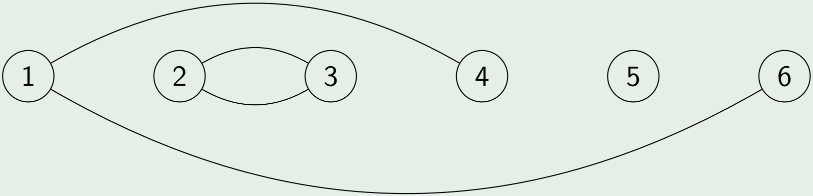

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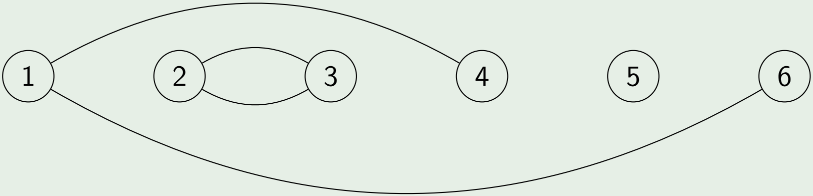

The meander above has one cycle and two paths.

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Theorem (Dergachev, Kirillov (2000))

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Theorem (Dergachev, Kirillov (2000))

Let $\mathfrak{g}$ be the seaweed algebra arising from λ, μ.

- *$\mathfrak{g}$ has a naturally associated meander $\mathcal{M}$.*
- *The index of $\mathfrak{g}$ depends only on the component structure of $\mathcal{M}$.*
- *If $\mathcal{M}$ has C cycles and P paths,*

$$ind(\mathfrak{g}) = 2C + P - 1.$$

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We construct the meander associated to $\lambda = (3, 3, 2)$ and $\mu = (4, 3, 1)$:

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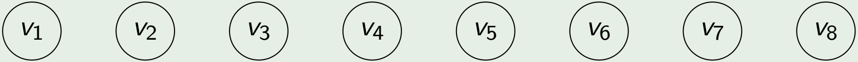

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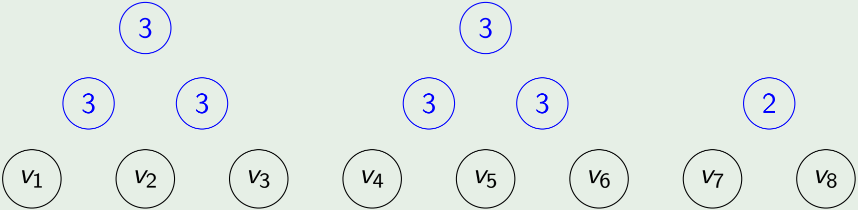

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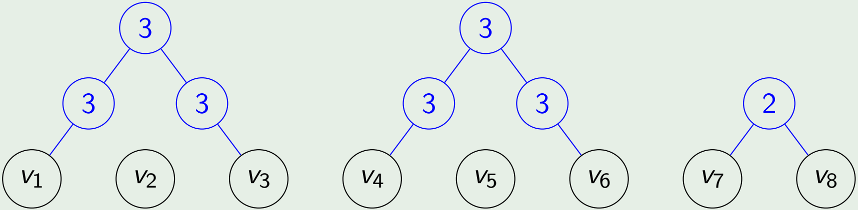

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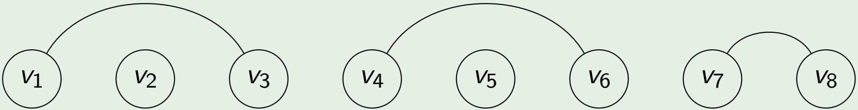

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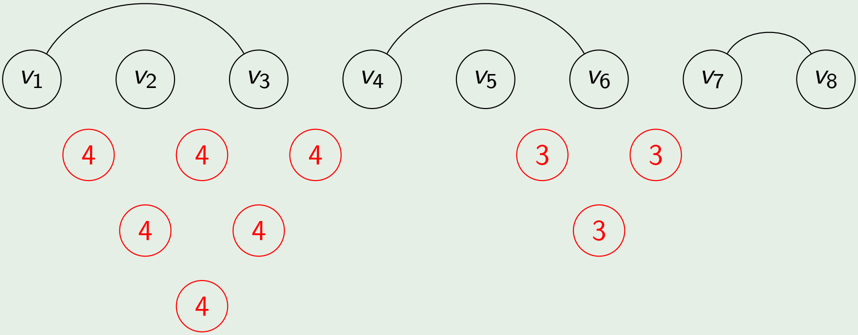

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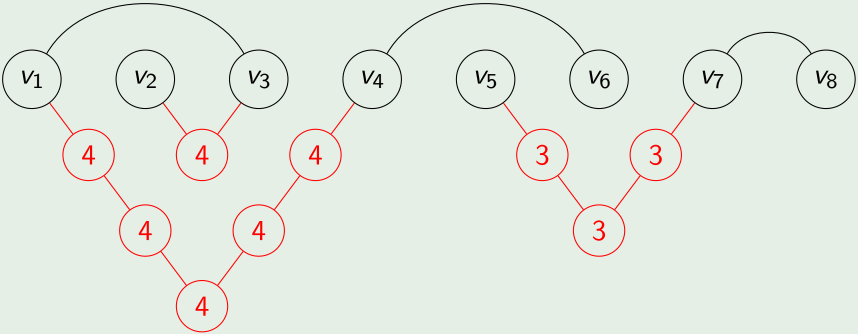

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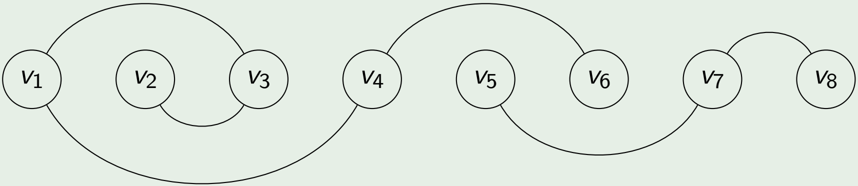

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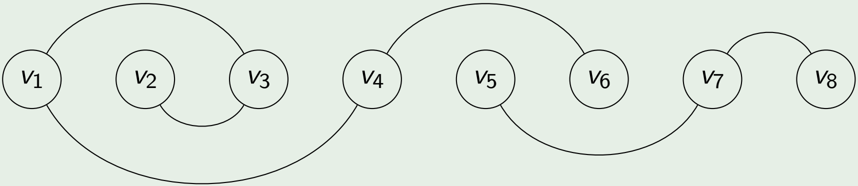

The associated seaweed algebra has index $2(0) + 2 - 1 = 1$.

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Fact (Coll, Mayers, Mayers)

$\text{ind}_{(1,1,\ldots,1)}(\lambda)$ is connected to 2-color partitions.

The Main Conjecture

Conjecture (Coll–Mayers–Mayers Conjecture)

Let $\mathcal{O} = \{\lambda : \text{odd parts}\}$ and $ind(\lambda) := ind_{(n)}(\lambda)$. If $o(n)$ (resp. $e(n)$) is the number of $\lambda \in \mathcal{O}$ of size n having odd (resp. even) index,

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$$G(q) = \left(q, -q^3; q^4\right)_\infty^{-1} = \sum_{n \geq 0} |o(n) - e(n)|\, q^n.$$

Conjecture is True Up to Sign

Theorem (Seo, Yee (2019))

We have

$$G(q) = \sum_{n \geq 0} (-1)^{\lceil \frac{n}{2} \rceil} \left(o(n) - e(n)\right) q^n.$$

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Remark

Later, $(-1)^{\lceil \frac{n}{2} \rceil}$ leads to eventually periodic signs for $o(n) - e(n)$.

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$$o(n) - e(n) = \begin{cases} N_0(n) - N_2(n) & \text{if } n \equiv 0 \pmod 2, \\ N_3(n) - N_1(n) & \text{if } n \equiv 1 \pmod 2, \end{cases}$$

where $N_k(n) := \#\{\lambda \in \mathcal{O} : \mathrm{op}(\lambda) \equiv k \bmod 4\}$.

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where $N_k(n) := \#\{\lambda \in \mathcal{O} : \text{op}(\lambda) \equiv k \bmod 4\}$.

- Using generating functions for $N_k(n)$,

$$G(q) = \sum_{n \geq 0} (-1)^{\lceil \frac{n}{2} \rceil} (o(n) - e(n)). \quad \square$$

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For $G(q) =: \sum_{n \geq 0} a(n) q^n$*, we have* $a(n) \geq 0$ *for all* $n > 2.4 \times 10^{14}$.

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Furthermore, as $n \to \infty$

$$a(n) \sim \frac{\pi^{1/4}\Gamma(1/4)}{2^{9/4}3^{3/8}n^{3/8}} I_{-3/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right) + (-1)^n \frac{\pi^{3/4}\Gamma(3/4)}{2^{11/4}3^{5/8}n^{5/8}} I_{-5/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right).$$

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Remark

- In principle, "Chern + Good Computer $\implies$ Coll–Mayers–Mayers"
- 2.4×10^5 took ≈ 9 hours.

Main Theorem

Theorem (C. (2021))

Using a different method, we get

$$a(n) = Chern's\ Formula + Error,$$

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Theorem (C. (2021))

Using a different method, we get

$$a(n) = \textit{Chern's Formula} + \textit{Error},$$

with error term small enough to show $a(n) \geq 0$ *for* $n > 4800$.

Proof by Circle Method

- By Cauchy's theorem,

$$a(n) = \frac{1}{2\pi i} \int_C \frac{G(q)}{q^{n+1}} dq$$

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- Since $G(q)$ is not modular, we use "Wright's variant" of the circle method to estimate $a(n)$.

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- This variation of Wright's method decomposes $a(n)$ into three integrals

$$a(n) = \frac{1}{2\pi i}\left(\int_{\tilde{C}} \frac{\tilde{G}(q)}{q^{n+1}}dq + \int_{\tilde{C}} \frac{G(q)-\tilde{G}(q)}{q^{n+1}}dq + \int_{C\setminus\tilde{C}} \frac{G(q)}{q^{n+1}}dq\right),$$

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- Proof uses effective estimates for $G(q)$. (Tedious!)

Asymptotics for $G(q)$

Theorem (C. (2021))

- *We have as* $z = x + iy \to 0$ *on the major arc* $0 < |y| < 30x < \pi$ *that*

$$G(q) \sim \tilde{G}(q) := \frac{2^{1/4} e^{\gamma/4}}{\sqrt{2\pi}\Gamma(1/4)} \cdot \frac{\exp\left(\frac{\pi^2}{48z}\right)}{z^{1/4}}.$$

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- *For* z *on the major arc with* $0 < x < \frac{\pi}{480}$,

$$\left| G(q) - \tilde{G}(q) \right| < \frac{23}{10} x^{1/4} \exp\left(\frac{\pi^2}{48x} + \frac{\sqrt{901}}{2} x + \frac{217}{5} x^2 \right).$$

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- $\tilde{G}(q)$ connects to the modified Bessel function $I_{-3/4}(z)$.
- $0 < x < \frac{\pi}{480}$ gives rise to $n > 4800$.

Asymptotics for $G(q)$

- By Euler–Maclaurin summation,

$$\sum_{n=a}^{b} f(n) \sim \int_a^b f(x)dx + \frac{f(b)+f(a)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(b) - f^{(2k-1)}(a) \right).$$

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- If $f(z) \sim \sum_{n=0}^{\infty} c_n z^n$ as $z \to 0$, we have for $0 < a \leq 1$ that

$$\sum_{n \geq 0} f\left((n+a)z\right) \sim \frac{1}{z} \int_0^{\infty} f(x)dx - \sum_{n=0}^{\infty} \frac{c_n B_{n+1}(a)}{(n+1)!} z^n.$$

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- Using classical Euler–Maclaurin, find explicit error terms $O\left(|z|^N\right)$.

Asymptotics for $G(q)$

- For $q = e^{-z}$,

$$\begin{aligned}\log\left(G(q)\right) &= \log\left(q;q^4\right)_\infty^{-1} + \log\left(-q^3;q^4\right)_\infty^{-1} \\ &= 4z\sum_{m\geq 1}\frac{e^{-mz}}{4mz\left(1-e^{-4mz}\right)} + 4z\sum_{m\geq 1}\frac{(-1)^m e^{-3mz}}{4mz\left(1-e^{-4mz}\right)}.\end{aligned}$$

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- Euler–Maclaurin shows $G(q)\sim\tilde{G}(q)$ and bounds $\left|G(q)-\tilde{G}(q)\right|$ via

$$\left|\log\left(G(q)\right) - \log\left(\tilde{G}(q)\right)\right| \leq \frac{1}{2}|z| + \frac{7}{5}|z|^2. \quad \square$$

Sketch of Minor Arc Bound

Theorem (C. (2021))

If $z = x + iy$ *satisfies* $0 < x < \frac{\pi}{480}$ *and* $30x \leq |y| < \pi$, *then*

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- The result would follow from

$$\mathrm{Re}\left(\log\left(G(q)\right)\right) = \sum_{m \ge 1} \frac{\cos(my)\, e^{-mx}}{m\left(1 + (-1)^m e^{-2mx}\right)} < \frac{1}{5x}.$$

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- This is proven by repeatedly "splitting off" early terms of the infinite sum along with bounds on the denominator arising from the Law of Cosines. □

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- The two other integrals making up $a(n)$ are error terms bounded by $E_{\text{maj}}(n)$ and $E_{\text{min}}(n)$ respectively.

Wrapping up the Circle Method

- $a(n) \geq 0$ follows from

$$\frac{\pi^{1/4}\Gamma\left(1/4\right)}{2^{9/4}3^{3/8}n^{3/8}} I_{-3/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right) > E_{\mathrm{main}}(n) + E_{\mathrm{maj}}(n) + E_{\mathrm{min}}(n).$$

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- Follows for $n > 4800$ by tedious estimations. □

Main Results

Conjecture (Coll–Mayers–Mayers)

Let $\mathcal{O} = \{\lambda : \text{odd parts}\}$ and $ind(\lambda) := ind_{(n)}(\lambda)$. If $o(n)$ (resp. $e(n)$) is the number of $\lambda \in \mathcal{O}$ of size n having odd (resp. even) index, then

$$G(q) := \left(q, -q^3; q^4\right)_\infty^{-1} = \sum_{n \geq 0} |o(n) - e(n)|\, q^n.$$

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We have

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Theorem (C.)

The coefficients of $G(q)$ *are non-negative.*

End of Talk

Thank You!