Partitions into parts simultaneously s-regular and t-distinct

Specialty Seminar in Partitions, q-Series, and Related Topics

William J. Keith, Michigan Technological University

Basic Definitions

The usual first theorem learned by a student in partition theory is

Theorem

The number of partitions of n into odd parts equals the number of partitions of n which parts are distinct.

Remark: In fact this was roughly *the* first theorem in partition theory, proved by Leonhard Euler in his work *De Partitio Numerorum*, which first systematically explored the concept.

Basic Definitions

This theorem was generalized by Sylvester and Glaisher, who proved that

Theorem

The number of partitions of n into parts not divisible by m equals the number of partitions of n which no part size appears m or more times.

We call the first type of partition *m-regular* and the second type *m-distinct*. Their generating function is

$$P_R(q) = P_D(q) = \prod_{k=1}^{\infty} \frac{1 - q^{mk}}{1 - q^k}.$$

Basic Definitions

Theorem

The number of m-regular partitions of n equals the number of m-distinct partitions of n.

Glaisher's proof was bijective:

Proof.

If in an m-distinct partition part jm^k appears $a_{j,k}$ times, $m \nmid j$, $a_{j,k} < m$, then write $a_{j,k} m^k$ appearances of j. Reverse by reading the m-ary expansion of the number of appearances of j. $\square$

Let us call the map from distinct to regular partitions ϕ_m and its reverse ${\phi_m}^{-1}$.

Simultaneous Conditions

Interesting maps + Interesting objects = Interesting fixed points

The fixed points of ϕ_m and ${\phi_m}^{-1}$ are precisely the partitions that satisfy both conditions for a given m.

Once we have brought up this notion, curiosity suggests a natural generalization: what can we do with the partitions simultaneously s-regular and t-distinct?

Regular and Distinct

The generating function for partitions which are s-regular and t-distinct is easy to write down: it is

$$P_{(s,t)}(q) = \sum_{n=0}^{\infty} p_{(s,t)}(n)q^n = \prod_{i=1}^{\infty} \frac{(1-q^{sk})(1-q^{tk})}{(1-q^k)(1-q^{stk})}.$$

Think of this as taking partitions into any size of part, removing from the allowable set those that are divisible by s, and then from this generating function limiting the number of appearances of each of the remaining parts. If we do so for all parts we over-limit by those divisible by s which we removed earlier.

Regular and Distinct (Tangent on Congruences)

Just a quick aside: $P_{(s,t)}(q)$ is an η-quotient, which is a well-understood class of functions. By work of Stephanie Treneer, it's known that all such functions are weakly holomorphic modular forms, and so we can hope that they will exhibit many congruences. For instance,

$$p_{(2,2)}(125n+99) \equiv 0 \pmod{5} \quad \text{(Rødseth)}$$

$$p_{(3,3)}(4n+2) \equiv 0 \pmod{2}$$

$$p_{(2,5)}(4n+3) \equiv 0 \pmod{2} \quad \text{and}$$

$$\sum_{n=0}^{\infty} p_{(2,5)}(4n+1)q^n \equiv \prod_{i=1}^{\infty}(1-q^{5k}) \pmod{2}.$$

All of these can be proved with techniques that are now fairly standard in the field.

Regular and Distinct

$$P_{(s,t)}(q) = \sum_{n=0}^{\infty} p_{(s,t)}(n) q^n = \prod_{i=1}^{\infty} \frac{(1-q^{sk})(1-q^{tk})}{(1-q^k)(1-q^{stk})}.$$

Notice that $P_{(s,t)}$ is symmetric in s and t. This means

Theorem

The number of partitions of n which are s-regular and t-distinct equals the number of partitions of n which are t-regular and s-distinct.

Regular and Distinct

If s and t are coprime, the map which realizes this combinatorially is ${\phi_s}^{-1}\phi_t$ – from an s-regular, t-distinct partition, rewrite parts of size jt^k as parts j appearing t^k times with any multiplicities needed, notice that parts of size j are still not divisible by s, and rewrite as js^k, which does not introduce divisibilities by t.

Example

Start with (50) as a 6-regular, 5-distinct partition:

$$(50) \underset{\phi_5}{\longrightarrow} (2, \ldots, 2) \underset{{\phi_6}^{-1}}{\longrightarrow} (12, 12, 12, 12, 2)$$

Regular and Regular

Is this a bijection? The intermediate set consists of partitions simultaneously s-regular and t-regular. Therefore they are suitable inputs for ${\phi_t}^{-1}$, so the sets are in bijection, and likewise in the other half of the map.

If we have s and t coprime, then $P_{s,t}(q)$ is indeed also the generating function for such partitions. Think of it as removing parts divisible by s, removing parts divisible by t, and then replacing those you overcounted.

Notice that for two conditions, distinctness is different from regularity: s-distinct t-distinct partitions are just $\min(s,t)$-distinct.

Non co-prime

If s and t are not coprime, then $P_{s,t}(q)$ is not the generating function for partitions simultaneously s-regular and t-regular; instead, that function is

$$\prod_{i=1}^{\infty} \frac{(1-q^{sk})(1-q^{tk})}{(1-q^{k})(1-q^{\mathrm{lcm}(s,t)k})}.$$

Non co-prime

Now the map ϕ_t preserves s-regularity, but ${\phi_s}^{-1}$ need not preserve t-regularity. For example, consider (50) as a 6-regular, 10-distinct partition:

Example

$$(50) \underset{(mod\,10)}{\longrightarrow} (5,5,5,5,5,5,5,5,5,5) \underset{(mod\,6)}{\longrightarrow} (30,5,5,5,5)$$

Non co-prime

During a visit to Michigan Tech by Bridget Tenner, of DePaul at the time, she conjectured that iteration of the map (1) was well-defined and (2) would realize the symmetry.

Example

$$(50) \underset{\phi_{10}}{\longrightarrow} (5, \ldots, 5) \underset{{\phi_6}^{-1}}{\longrightarrow} (30, 5, 5, 5, 5)$$

$$\underset{\phi_{10}}{\longrightarrow} (5, 5, 5, 5, 3, \ldots, 3) \underset{{\phi_6}^{-1}}{\longrightarrow} (18, 5, 5, 5, 5, 3, 3, 3, 3)$$

Non co-prime

This conjecture is interesting and, as I'll show later, still worthy of study, but as stated does not hold.

Consider $(108, 18, 18, 18, 18)$ as a 10-regular, 6-distinct partition. Applying ϕ_6 yields $(3, \ldots, 3)$, the part appearing 60 times. Then ${\phi_{10}}^{-1}$ yields $(30, 30, 30, 30, 30, 30)$. This is neither 6-regular nor 6-distinct, so we are not done, but it is not in the domain of ϕ_6.

Realizing the symmetry

So the question is still before us: what is a map that realizes the symmetry?

This will be the subject of the next part of the talk. Before I go on, let's pause and ask for any questions!

Realizing the symmetry

We'll begin with a simple case and work our way up.

We begin with an s-regular, t-distinct partitions. Let t be arbitrary, but say that $s = p$ is a prime, presumably one that divides t. Write $t = p^r k$, with $p \nmid k$.

Suppose that part j appears $a + Ck$ times, $0 \leq a < k$. It is necessarily true that $C < p^r$.

Realizing the symmetry: s prime

Collect the parts j appearing a times. Since $a < k$, these form a partition into parts not divisible by p, appearing less than k times, and $p \nmid k$, so we have a p-regular, k-distinct partition with k and p coprime.

Thus we can apply ${\phi_p}^{-1}\phi_k$ and obtain a partition which is k-regular (hence t-regular) and p-distinct.

Write $C = c_0 p^0 + c_1 p^1 + \cdots + c_{r-1} p^{r-1}$, each $0 \leq c_i < p$. Now for each term add to the target partition c_i appearances of $p^i kj$. Since j is p-regular, these parts are divisible by k but not $p^r k = t$, so the result is t-regular, and $c_i < p$, so the result is p-distinct.

Realizing the symmetry: s prime

To show that the map is reversible, consider parts divisible and not divisible by k separately; the first form a k-regular, p-distinct partition. For those that are divisible by k, divide them by k and by whatever power p^i appears in their factorization; replace them with the resulting divisor appearing $p^i k$ times.

The result is p-regular and t-distinct. Notice that parts j can appear in multiple ways, but different sources yield appearances in different powers of p.

Realizing the symmetry: s prime

Before we generalize s, let's look at a quick example. Let $\lambda = (9, 9, 9, 9, 9, 9, 9, 5, 5)$ be a 2-regular, 12-distinct partition. We have $t = 2^2 \cdot 3$, so $k = 3$.

The part 5 appears twice, or $2 + (0) \cdot 3$ times. We treat $(5, 5)$ as a 2-regular, 3-distinct partition, and apply ${\phi_2}^{-1}\phi_3$. The first map ϕ_3 does nothing, and ${\phi_2}^{-1}$ produces a distinct part of size 10.

The part 9 appears $7 = 1 + (0 \cdot 2^0 + 1 \cdot 2^1) \cdot 3$ times. The one appearance of 9 is treated as the partition (9), and we have ${\phi_2}^{-1}\phi_3(9) = {\phi_2}^{-1}(1, 1, 1, 1, 1, 1, 1, 1, 1) = (8, 1)$, and the $(1 \cdot 2^1) \cdot 3$ appearances of 9 becomes 1 appearance of $9 \cdot 2^1 \cdot 3 = 54$.

The final partition is $(54, 10, 8, 1)$, a 12-regular, 2-distinct partition.

Realizing the symmetry: s prime

To reverse from $(54, 10, 8, 1)$, we now consider a 12-regular, 2-distinct partition, so t is prime. We calculate $k = 3$.

The distinct parts not divisible by 3 are $(10, 8, 1)$, and if we apply ${\phi_3}^{-1}\phi_2$ we get

$${\phi_3}^{-1}\phi_2(10, 8, 1) = {\phi_3}^{-1}(5, 5, 1, 1, 1, 1, 1, 1, 1, 1, 1) = (9, 5, 5).$$

The part 54, divisible by 3, is $2^1 \cdot 3 \cdot 3^2$, and hence we write another $2^1 \cdot 3$ appearances of 9. This regains our original partition.

Realizing the symmetry: $s = p^j$

Suppose now that s is a prime power, $s = p^j$. We break into cases:

1. Collect parts not divisible by p and apply the previous map. We obtain a t-regular, p-distinct partition.
2. Collect parts divisible by p but not p^2. Divide by p and apply the previous map. Multiply all resulting frequencies by p. The result is t-regular and p^2-distinct with frequency divisible by p.
3. Collect parts divisible by p^2 but not p^3, divide by p^2, apply the previous map, and multiply the resulting frequencies by p^2.
4. Etc.

Realizing the symmetry: $s = p^j$

Parts may appear repeatedly in various steps but the sum of their number of appearances is less than p^j, and by writing the p-ary expansion of the number of appearances, we can break up the partition into the images of the various steps, and reverse the map as before.

Realizing the symmetry: general s

Finally, what if $s = {p_1}^{e_1} {p_2}^{e_2} \dots {p_b}^{e_b}$?

Any given part is divisible by possibly some but not all of the p_i. We run through a similar recursion process, now with each prime.

Suppose a part is not divisible by ${p_1}^{e_1}$. The collection of such parts forms a ${p_1}^{e_1}$-regular, t-distinct partition, and we apply the previous map to obtain a t-regular, ${p_1}^{e_1}$-distinct partition.

Realizing the symmetry: general s

Next suppose a part is divisible by ${p_1}^{e_1}$ but not ${p_2}^{e_2}$. Divide out the ${p_1}^{e_1}$ and treat the collection of such parts as a ${p_2}^{e_2}$-regular, t-distinct partition.

Apply the previous map to obtain a t-regular, ${p_2}^{e_2}$-distinct partition, and multiply the frequency of appearance by ${p_1}^{e_1}$. We now have a t-regular partition with frequencies of appearance divisible by ${p_1}^{e_1}$ but at most ${p_1}^{e_1}({p_2}^{e_2} - 1)$.

Gathered with the parts from the previous step, we obtain a t-regular partition in which frequencies can be anything below ${p_1}^{e_1}{p_2}^{e_2}$.

Realizing the symmetry: general s

Repeat with the remaining primes and we eventually obtain a t-regular, s-distinct partition, and the desired map is constructed. □

Questions for thought

What partitions are the fixed point of this symmetry?

For s prime, the parts appearing less than k times (not counting multiples of k) would be an s-regular, k-distinct partition, which is fixed only if also k-regular and s-distinct. The parts appearing Ck times would be multiplied by k so they are never fixed if they exist.

Thus for s-regular, t-distinct partitions with s prime, being fixed under our map actually requires k-distinctness when $t = s^j k$.

What is the fixedness condition for more complicated s? What is the generating function for these partitions, and does it have any nice properties?

Questions for thought

The generating function $p_{2,2}(q)$ is very special: it counts partitions into distinct odd parts, which are in bijection with self-conjugate partitions and thus share the parity of the partition function $p(n)$.

The functions $P_{m,m}(q)$ can share other congruences with $P(q)$: for instance, $p_{5,5}(5n+4) \equiv 0 \pmod 5$, as is easily seen since the coefficients of $p_{5,5}(n)$ can be written as a recurrence in $p(n-5j)$.

Questions for thought

However, the partitions of $5n+4$ that are 5-regular and 5-distinct are a nice subset of all partitions of $5n+4$, and the reverse recurrence can also be written: ergo, if one could find a simple combinatorial proof of $p_{5,5}(5n+4) \equiv 0 \pmod{5}$ it would immediately imply the same property for $p(n)$.

(Note: Dyson's rank and crank are not equidistributed on this class mod 5. Is another statistic?)

Questions for thought

Under what conditions does $\left(\phi_s{}^{-1}\phi_t\right)^{\ell}$ successfully realize the symmetry? Does it suffice to have $s < t$? No: a counterexample is $(5,5,5,5,5,5,1,1,1,1,1,1)$ as a 6-regular, 10-distinct partition.

When it does work, is it the map we just constructed? What ℓ is required, and can one easily tell from the partition?

References

Thank you!

1. Keith, W. J. Partitions into parts simultaneously regular, distinct, and/or flat. https://arxiv.org/abs/1911.04755
2. Rødseth, Ø. Dissections of the generating functions of $q(n)$ and $q_0(n)$, Arbok Univ. Bergen Mat.-Natur. 1969, No. 12 (1970), 3-12. MR0434959 (55:7922)
3. Treneer, S. Congruences for the coefficients of weakly holomorphic modular forms. Proc. Lond. Math. Soc. 93, 304-324, 2006