Fixed points among hook lengths

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Outline

1. Background
2. Part size equal to multiplicity
3. The truncated pentagonal number theorem
4. If time, brief observations
5. Further questions

Introduction

In September 2023 in this Seminar, Brian Hopkins presented "Partition Fixed Points: Connections, Generalizations, and Refinements," on work partially joint with James Sellers.

That talk took an idea of Blecher & Knopfmacher, fixed points in integer partitions, and connected it to a wide array of combinatorial quantities on partitions: Frobenius symbols, the crank, the mex, and more.

A *fixed point* in a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_j)$, written in decreasing order $\lambda_i \geq \lambda_{i+1}$, is a part $\lambda_i = i$. Among the partitions of 5, three have fixed points - $\{32, 221, 11111\}$ - while four do not - $\{5, 41, 311, 2111\}$.

Introduction

It's clear that, when partitions are written in descending order, a given partition can have at most 1 fixed point.

Among the theorems Brian and James proved in their paper were:

Theorem

(Conjectured by Blecher & Knopfmacher) For $n > 2$, there are more partitions of n without a fixed point than with a fixed point. (H & S) In fact the excess equals the number of crank 0 partitions.

Theorem

For given n, the sum over all i of the number of partitions of $n + 1 - i$ with a fixed point at place i equals the number of partitions of n with no part equal to the size of their Durfee square.

Introduction

In the audience at that talk, David Hemmer of Michigan Tech pointed out that another descending sequence in a partition is its first-column hook lengths, or *beta numbers*, which are especially of interest to representation theorists.

4	2
3	1
1	

Here the sequence of first-column hook lengths is $(4, 3, 1)$. As a strictly decreasing sequence, it is meaningful to consider fixed points among the beta numbers.

Introduction

He suggested that this would be fruitful to consider, so he, Brian, myself, and my graduate student Philip Cuthbertson started working on the project.

It turns out there are several exciting identities associated with these fixed points! This will be the topic of my talk today.

Main Theorems

One of our main theorems is:

Theorem

The number of partitions of n having a fixed first-column hook is equal to the number of times in all partitions of n that a part of size i appears with multiplicity exactly i.

Notice the latter can happen multiple times in a single partition.

Example

Partitions of 5 are $\{5, 41, \mathbf{32}, \mathbf{311}, 221, 2111, \mathbf{11111}\}$. Their beta-numbers are $\{5, 51, \mathbf{42}, \mathbf{521}, 431, 5321, \mathbf{54321}\}$ respectively. The three bold partitions have a fixed hook. Multiplicity equaling part size occurs for 41 once, and for 221 twice.

Main Theorems

We also have an intriguing connection with work of Andrews and Merca on the truncated pentagonal number theorem:

Theorem

Fix $k > 0$. The number of times in all partitions of $n - \binom{k}{2}$ for which some $\lambda_i = k$, with $h_{i,1} = i - 1$, equals $M_k(n)$, the number of partitions of n in which the smallest size not appearing is k and the number of parts larger than k is greater than the number of parts smaller than k.

Andrews and Merca show that this latter quantity is also $(-1)^{k+1}$ times the sum that arises when one takes the first $2k$ terms of the *pentagonal number recurrence* for $p(n)$:

$$M_1(n) = p(n) - p(n-1)\,,\ M_2(n) = p(n) - p(n-1) - p(n-2) + p(n-5)\,, \ldots .$$

Fixed hooks

The main object of interest is an *h-fixed hook*: a (first-column) hook arising from the part in place s which is of hook length $s+h$, where $h \in \mathbb{Z}$. Without number, a *fixed hook* is a 0-fixed hook, which was our initial object.

9	7	6	5	2	1
6	4	3	2		
5	3	2	1		
1					

Here the first-column hook lengths $(9, 6, 5, 1)$ are respectively 8-fixed, 3-fixed, 2-fixed, and -3-fixed.

Fixed hooks

To write down our generating function, we'll need the following notation, all of which is standard:

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k),\ (a;q)_n = \frac{(aq^n;q)_\infty}{(a;q)_\infty},$$
$$(q)_n = (q;q)_n,\ \begin{bmatrix} A \\ B \end{bmatrix}_q = \frac{(q)_A}{(q)_B(q)_{A-B}}.$$

Now $\frac{1}{(q)_n}$ is the generating function for partitions with parts at most n, and $\begin{bmatrix} A \\ B \end{bmatrix}_q$ is the generating function for partitions in the $(A-B)\times B$ box, i.e., having at most $A-B$ parts, all of which are of size at most B, or vice versa.

Fixed hooks

The following diagram is a primary tool in many of our arguments. We have an h-fixed hook arising from a part of size k at place s.

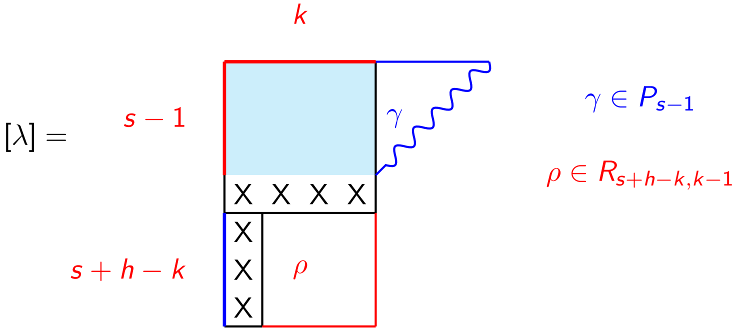


Figure: λ with an h-fixed hook $h_{1,s} = s + h$ at part $\lambda_s = k$.

Fixed hooks

From inspection and standard combinatorial arguments, this gives:

Theorem

The generating function for the number of partitions of n with an h-fixed hook arising from a part of size k is

$$\sum_{s=k-h}^{\infty} q^{(k+1)(s-1)+h+1} \begin{bmatrix} s+h-1 \\ k-1 \end{bmatrix}_q \frac{1}{(q)_{s-1}}$$

$$= q^{h+1-\binom{k}{2}} \sum_{s=k-h}^{\infty} q^{(k+1)(s-1)+\binom{k}{2}} \begin{bmatrix} s+h-1 \\ k-1 \end{bmatrix}_q \frac{1}{(q)_{s-1}}.$$

0-Fixed hooks equals "size equals multiplicity"

So consider 0-fixed hooks at all possible places s arising from parts of any possible size k, and sum: the generating function for the number of fixed hooks is

$$\sum_{s=1}^{\infty}\sum_{k=1}^{\infty} q^{(s-1)k}\frac{1}{(q)_{s-1}}\cdot q^{s-k}\begin{bmatrix} s-1 \\ s-k \end{bmatrix}_q$$

$$= \sum_{s=1}^{\infty}\sum_{k=1}^{\infty} q^{(s-1)k+s-k}\cdot\frac{1}{(q)_{k-1}(q)_{s-k}}.$$

0-Fixed hooks equals "size equals multiplicity"

Simplify the denominator a bit at the cost of the numerator, setting $T = k - 1$, $j = s - k$: our function is

$$\sum_{T=0}^{\infty} \sum_{j=0}^{\infty} \frac{q^{(T+1)^2+(T+2)j}}{(q)_T (q)_j} = \sum_{T=0}^{\infty} \frac{q^{(T+1)^2}}{(q)_T} \sum_{j=0}^{\infty} \frac{q^{(T+2)j}}{(q)_j}.$$

We now use the identity $\sum_{j=0}^{\infty} \frac{q^{(T+2)j}}{(q)_j} = \frac{1}{(q^{T+2};q)_\infty}$ to get that our generating function is

$$\sum_{T=0}^{\infty} \frac{q^{(T+1)^2}}{(q)_T (q^{T+2}; q)_\infty} = \sum_{T=0}^{\infty} \frac{q^{(T+1)^2}(1 - q^{T+1})}{(q)_\infty}.$$

This is exactly the generating function counting partitions in which some part size indexed by $T + 1$ appears with multiplicity exactly $T + 1$, and the theorem is proved.

0-Fixed hooks equals "size equals multiplicity"

In fact, because the index $T + 1$ is exactly part size k, we have the following refinement:

Theorem

The number of partitions $\lambda \vdash n$ which contain the part k with multiplicity k is the same as the number of partitions $\mu \vdash n$ for which there is a fixed hook $h_{s,1} = s$ with $\mu_s = k$.

0-Fixed hooks equals "size equals multiplicity"

Without going in to detail, we'll note that the theorem can be proved bijectively as well. One requires a bijection for the identity

$$\frac{1}{(q)_a} \cdot \frac{1}{(q)_b} = \frac{1}{(q)_{a+b}} \begin{bmatrix} a+b \\ a \end{bmatrix}_q .$$

We found one on Stackexchange from one "Splutterwit." If you know them, give us a lead!

Truncated pentagonal number theorem

Some quick context: Euler gave the expansion of the infinite product

$$(q;q)_\infty = 1 - q - q^2 + q^5 + q^7 - \cdots = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{n}{2}(3n-1)},$$

and the fact that $\frac{1}{(q;q)_\infty} = \sum_{n=0}^{\infty} p(n)q^n$, the generating function for the partition numbers. Now the relation

$$\frac{(q)_\infty}{(q)_\infty} = 1$$

yields the *pentagonal number recurrence* for $p(n)$, namely $p(n) = 0$ for $n < 0$, $p(0) = 1$, and for $n > 0$,

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - p(n-12) + \cdots = 0.$$

Truncated pentagonal number theorem

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - p(n-12) + \cdots = 0$$

It's not too difficult to see that signs alternate in pairs.

Since the partition function $p(n)$ is strictly increasing for $n > 1$, it's clear that if you truncate this sum after an *odd* number of terms, you will always get a nonnegative or nonpositive value depending on whether the number of terms is 1 or 3 mod 4 respectively, since remaining value to be added to get 0 is either negative or positive.

It's much less obvious what the sign of truncation after an even number of terms is, and whether it is even consistently one or the other for a fixed number of terms.

Truncated pentagonal number theorem

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - p(n-12) + \cdots = 0$$

After two terms:

$$p(n) - p(n-1)$$

This counts a combinatorial set: it is the number of partitions with no 1, for to any partition of $n-1$ we simply append a 1, and thereby get the subset of partitions of n that have at least one 1. After four terms:

$$p(n) - p(n-1) - p(n-2) + p(n-5)$$

Much less clear!

Truncated pentagonal number theorem

What Andrews and Merca showed is that, truncating after $2k$ terms, we do consistently get a negative or positive value: the truncated sum is $(-1)^{k+1}M_k(n)$, where we have

$$\mathcal{M}_k = \sum_{n=0}^{\infty} M_k(n) q^n = \sum_{n=k}^{\infty} \frac{q^{\binom{k}{2}+(k+1)n}}{(q)_n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q .$$

Furthermore, we can use the various parts of this generating function to see that this is counting the number of partitions of n in which the parts 1 through $k-1$ appear and the part k does not (i.e. the *mex* is k), and the number of parts larger than k is greater than the number of parts smaller than k.

Truncated pentagonal number theorem

But this generating function is very similar to the function we wrote down for fixed hooks! We can quickly get the following theorem:

Theorem

The number of times in all partitions of n that an h-fixed hook arises from a part of size k equals the number of partitions of $n + \binom{k}{2} - (h+1)$ with mex k where $(h + 1 +$ the number of parts larger than $k)$ is greater than the number of parts less than k.

Truncated pentagonal number theorem

Proof:

The constant term outside the sum is the shift given.

Let there be $s-1$ parts of size at least $k+1$, generated by $\frac{q^{(k+1)(s-1)}}{(q)_{s-1}}$. So we are allowed at most $s-1+h$ parts less than k. Since the mex is k, there is at least one part each of sizes 1 through $k-1$, giving $q^{\binom{k}{2}}$.

Let as many as $s+h-k$ further parts of size at most $k-1$ be appended, generated by $\begin{bmatrix} s+h-1 \\ k-1 \end{bmatrix}_q$.

Truncated pentagonal number theorem

And now in the case $h = -1$, the relation is exact. The $h + 1$ term vanishes and the generating function becomes precisely $q^{-\binom{k}{2}}\mathcal{M}_k(q)$. We have the following corollary:

Corollary

The quantity $M_k(n)$ can also be interpreted as the number of times in all partitions of $n - \binom{k}{2}$ that a -1-fixed hook arises from a part of size k.

Truncated pentagonal number theorem

This can also be given a bijective proof: let λ be a partition of n with a -1-fixed hook in position s arising from a part of size k. Thus $h_{1,s}(\lambda) = s-1$, $\lambda_s = k$, and λ has $2s-k-1$ nonzero parts.

Delete the part $\lambda_s = k$ and subtract one from the remaining $s-k-1$ nonzero parts $\lambda_{s+1}, \lambda_{s+2}, \ldots, \lambda_{2s-k-1}$ below it. Then add one to each of the first $s-1$ parts. Now append a part of every size from 1 to $k-1$.

Thus μ is a partition of $n + \binom{k}{2}$, has minimal excludant k, and there are $s-1$ parts larger than k and at most $s-2$ parts less than k as desired.

Additional observations

By summing we can write some generating function identities which might not be obvious on their own.

Theorem

The generating function for the number of partitions of n with an h-fixed hook arising from a hook of size t in any place is

$$\sum_{l=1}^{k} \frac{q^{t+l(t-h-1)}}{(q)_{t-h-1}} \begin{bmatrix} t-1 \\ l-1 \end{bmatrix}_q .$$

Additional observations

Sum this over t and we get

Theorem

The generating function for the number of partitions of n with an h-fixed hook is

$$\sum_{l=1}^{\infty} q^{l(-h-1)} \sum_{k=l}^{\infty} \frac{q^{(l+1)k}}{(q)_{k-h-1}} \begin{bmatrix} k-1 \\ l-1 \end{bmatrix}_q .$$

One notes that in the case $h = -1$, the inner sum becomes precisely $q^{-\binom{l}{2}} \mathcal{M}_l(q)$, and so we are getting a shifted sum over all of the $\mathcal{M}_l$. Is this meaningful?

Additional observations

And finally, if we fix the hook length k instead, we get the generating function for first-column k-hooks:

Theorem

The generating function for the number of first column k-hooks in all partitions of n is

$$\frac{q^k}{(q^k;q)_\infty} \sum_{l=1}^{k} \frac{1}{(q)_{k-l}}.$$

This is a pretty simple-looking generating function: could there be a more direct bijective proof?

Further questions

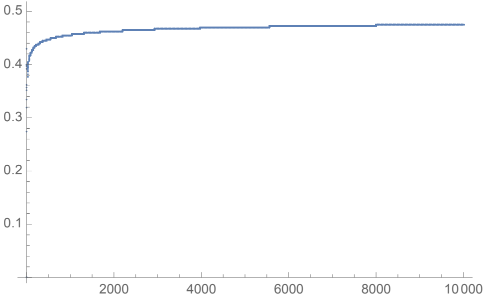
The connection between the pentagonal number theorem and the first-column hook lengths is decidedly intriguing. So far, we haven't been able to find more combinatorial consequences of this relationship, or extend it to something else, but the similarity between the two generating functions seems to suggest that something deeper connects the two. Is that true, and if so, what?

Further questions

Can we answer a question similar to Hopkins and Sellers on how many partitions have a fixed hook compared to the number of all partitions? The number of crank 0 partitions approaches 0% of the number of partitions of n - is the same holding true for the difference between the number of partitions with and without a fixed hook?

Further questions

Here's the computational evidence, the ratio of partitions with a fixed hook to all partitions of n, up to $n = 10000$:



There are a few more theorems that can be found in our submitted paper, at arXiv:2401.06254 .

Thank you!