Ramanujan-Kolberg identities, regular partitions, and multipartitions

Seminar in Partition Theory, q-Series and Related Topics

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Outline

1. Motivation: the parity problem
2. Ramanujan-Kolberg identities
3. First branch: Judge and Chen
4. Second branch: m-regular partitions, m odd
5. Grafting the branches: m-regular partitions, m even
6. Open questions

The parity of $p(n)$

The parity of the partition numbers is one of the big open questions in partition theory. If we define the *arithmetic density* of a property T of an integer-indexed sequence $\{f(n)\}_{n=n_0}^{\infty}$ to be

$$\delta(T) = \lim_{n\to\infty} \#\{k < n | T \text{ holds for } f(k)\},$$

then the *Parkin-Shanks Conjecture* or, colloquially, the *Fifty-Fifty Conjecture* is that

Conjecture

The densities $\delta(p(n) \equiv 0 \pmod 2)$ *and* $\delta(p(n) \equiv 1 \pmod 2)$ *both exist and equal 1/2.*

The parity of $p(n)$

We are very far from settling this conjecture. The best bounds for the cardinalities of the even and odd partition numbers are from work of Bellaïche and Nicolas:

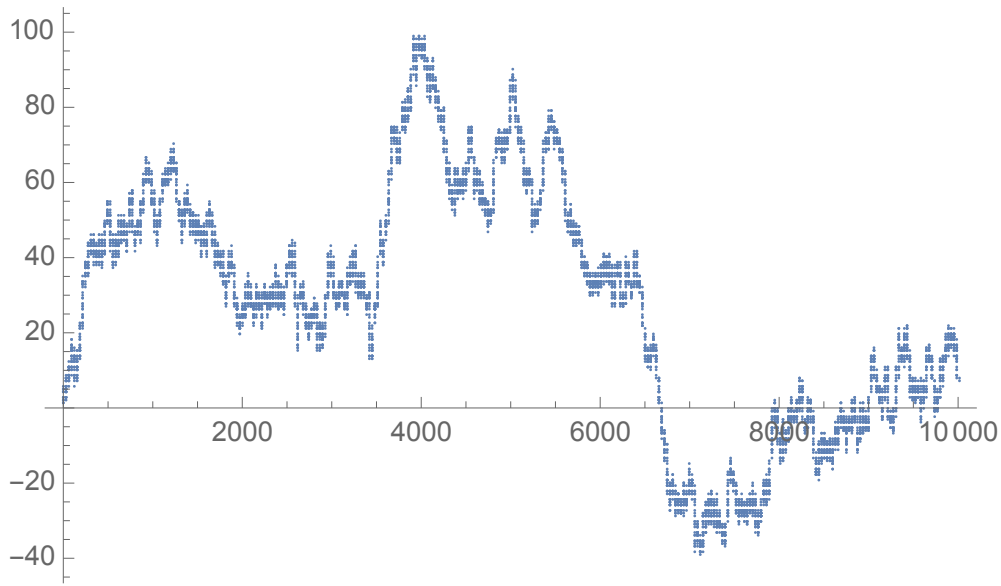
Theorem

$\#\{n < x | p(n) \text{ is even } \} > c\sqrt{x} \ln \ln x$ *and*
$\#\{n < x | p(n) \text{ is odd } \} > c\frac{\sqrt{x}}{(\ln x)^{7/8}}$.

So we do not even know if the densities, should they exist, are nonzero.

The parity of $p(n)$

If we simply look at the residues of the partition numbers mod 2, there seems to be very little pattern. For instance, here is the running excess of the number of odd partition numbers less than n over the number of even partition numbers, plotted up to 10000:



The parity of $p(n)$

This is in contrast to the residue of $p(n)$ modulo primes other than 2 or 3, which behaves nicely at least some of the time in the sense that we have some extremely regular patterns, in the form of the Ramanujan congruences and their relatives:

$$p(5n+4) \equiv 0 \pmod{5}$$
$$p(7n+5) \equiv 0 \pmod{7}$$
$$p(11n+6) \equiv 0 \pmod{11}$$

The parity of $p(n)$

But for 2 and 3, we now know that there is no such arithmetic progression, thanks to work of Cristian-Silviu Radu:

Theorem

(Radu) There exists no nonzero integer pair (A, B) *such that* $p(An + B) \equiv 0 \pmod 2$ *or* $\pmod 3$ *for all* $n \geq 0$.

So the intuition is that the parity of $p(n)$ is "essentially random," in that the limiting probability of an even or odd value is one-half, and any algorithm to predict this parity is unlikely to be an order of magnitude easier than simply calculating $p(n)$.

The parity of $p(n)$

Given all of this, it seems like it would be interesting to

1. find any pattern at all in the parity of $p(n)$, and
2. find statements involving $p(n)$ whose truth or falsity is equivalent to the parity conjecture or even a weaker bound.

Multipartitions

The same kind of behavior seems to hold for multipartitions, or t-colored partitions. A t-tuple of partitions summing overall to n is a *t-multipartition* of n, the number of which we will denote by $p_t(n)$. Denote $f_i = \prod_{k=1}^{\infty}(1 - q^{ki})$. Then the generating function of t-multipartitions is ${1/f_1}^t$.

Multipartitions

Denote by δ_t the density of the odd coefficients of the t-multipartitions. Similar computational evidence suggests

Conjecture

δ_t exists and equals $\frac{1}{2}$, for any odd positive integer t. Equivalently, if $t = 2^k t_0$ with $t_0 \geq 1$ odd, then δ_t exists and equals 2^{-k-1}.

Furthermore, like the partition function, there appear to be no arithmetic progressions in the multipartitions, for t odd, that are identically 0 mod 2.

Ramanujan-Kolberg identities

A few years ago Fabrizio Zanello and I proved a set of identities relating the parity of arithmetic progressions in partitions and multipartitions (from here out all congruences are mod 2 unless otherwise stated):

Theorem

The congruence

$$q\sum_{n=0}^{\infty} p_t(an+b)q^n \equiv \frac{1}{{f_1}^{at}} + \frac{1}{{f_a}^{t}}$$

holds for the following triples (a, b, t)*: (5,4,1), (7,5,1), (11,6,1), (13,6,1), (17,5,1), (19,4,1), (23,1,1), (3,2,3), (5,2,3), (7,1,3), (5,0,5), and (3,0,9).*

Ramanujan-Kolberg identities

And a three-term identity:

Theorem

The congruence

$$q^2 \sum_{n=0}^{\infty} p_t(a^2 n + b) q^n \equiv \frac{1}{{f_1}^{a^2 t}} + \frac{1}{{f_a}^{at}} + \frac{q}{{f_1}^{t}}$$

holds for the triples $(a, b, t) = (3, 8, 3)$ *and* $(5, 24, 1)$.

Ramanujan-Kolberg identities

Cristian-Silviu Radu named such identities, relating arithmetic progressions in the partition function and eta-quotients, *Ramanujan-Kolberg identities* after Ramanujan's "most beautiful identity"

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{{f_5}^5}{{f_1}^6}$$

and Kolberg's further study of these. He produced an algorithm for verifying them, which was automated by Nicolas Smoot; Zanello and I proved some of our identities with classical q-series dissections and some using Radu's machinery.

Consequences

For the remainder of the talk all arguments will assume that the densities δ_t exist. Then these theorems imply the following relationships among the δ_t:

Corollary

If $\delta_t > 0$, then $\delta_1 > 0$ for $t \in \{5, 7, 11, 13, 17, 19, 23, 25\}$ and $\delta_t > 0$ implies $\delta_r > 0$ for (t, r) any of the pairs (27,9), (9,3), (27,3), (25,5), (15,3), (21,3).

A curious fact here is that we never seem to get $\delta_3 > 0$ implying $\delta_1 > 0$. The families of identities and arithmetic progressions involving the two seem to be strangely separate.

Consequences

The argument for each of the corollary clauses is similar, so let's go through the first case, $\delta_5 > 0$ implies $\delta_1 > 0$. We take one of the cases of the main theorem, namely (5,4,1):

$$q\sum_{n=0}^{\infty} p(5n+4)q^n \equiv \frac{1}{{f_1}^5} + \frac{1}{f_5}.$$

Now suppose $\delta_5 > 0$ and $\delta_1 = 0$. Then for sufficiently large x, $\#\{n \leq x | p_5(n) \text{ is odd } \} = \delta_5(x) + o(x)$, while the number of odd coefficients up to x of $1/(q^5;q^5)_\infty = \sum_{n=0}^{\infty} p(n)q^{5n}$ is $\#\{n \leq x/5 | p(n) \text{ is odd } \}$ is $o(x)$. Then summing we find that the number of odd coefficients up to x of $\sum_{n=0}^{\infty} p(5n+4)q^n$ is $\delta_5(x) + o(x)$. But then $\delta_1 \geq \delta_5/5 > 0$, a contradiction.

Consequences

But you can get much stranger consequences than that.

Corollary

If $\delta_1 = 1$, then $\delta_5 = 4/5$, with density zero for the odd coefficients of the series $\sum_{n=0}^{\infty} p_5(5n)q^{5n}$ and density 1 among all other coefficients.

Observe the $(5, 4, 1)$ identity again:

$$q\sum_{n=0}^{\infty} p(5n+4)q^n \equiv \frac{1}{{f_1}^5} + \frac{1}{f_5}.$$

If $\delta_1 = 1$, then the left-hand side has density 1. On the right-hand side, the coefficients of q^{5n} in the second term are odd with density 1, and so the same coefficients in the first term have to be odd with density zero; meanwhile, outside of this progression, the terms must be odd with density 1.

Consequences

Ergo, on the assumption that all the densities involved exist:

Corollary

If the odd density of $p_5(n)$ is not distributed in this convoluted way, then $\delta_1 < 1$, and thus the even coefficients of $p(n)$ have positive density.

First branch: Judge and Chen

Fabrizio's doctoral student Samuel Judge extended the set of known identities in his doctoral thesis. Among those he found were:

$$q^2 \sum_{n=0}^{\infty} p(29n+23)q^n \equiv \frac{1}{{f_1}^{29}} + \frac{q}{{f_1}^{5}} + \frac{1}{f_{29}}$$

$$q^2 \sum_{n=0}^{\infty} p(35n+19)q^n \equiv \frac{1}{{f_1}^{35}} + \frac{q}{{f_1}^{11}} + \frac{1}{f_{35}} + \frac{1}{{f_7}^{5}} + \frac{1}{{f_5}^{7}}$$

$$q^3 \sum_{n=0}^{\infty} p(49n+47)q^n \equiv \frac{1}{{f_1}^{49}} + \frac{q}{{f_1}^{25}} + \frac{q^2}{f_1} + \frac{1}{{f_7}^{7}}.$$

First branch: Judge and Chen

This was enough for him to state a broad conjecture:

Conjecture

For odd integers a and t, with $3|t$ if $3|a$, let $b \equiv \frac{t}{3} \cdot 8^{-1}$ if $3|t$ and $b \equiv 24^{-1} \pmod{a}$ if not, and $k = \lceil \frac{t(a^2-1)}{24a} \rceil$. Then, in $\mathbb{Z}_2[[q]]$,

$$q^k \sum_{n=0}^{\infty} p_t(an+b)q^n \equiv \sum_{d|a} \sum_{j=0}^{\lfloor k/d \rfloor} \frac{\epsilon_{a,d,j}^t q^{dj}}{{f_d}^{at/d-24j}}$$

with $\epsilon_{a,d,j}^t \in \{0,1\}$, and $\epsilon_{a,1,0}^t = 1$ (i.e., the "largest" term appears) and $\epsilon_{a,d,j}^t = 0$ for $at/d - 24j < 0$ (i.e., no negative powers appear).

First branch: Judge and Chen

Just last year, Shi-Chao Chen was able to prove this for a any power of an odd prime at least 5, or for 3 itself. Chen describes the main idea of their proof to be application of the Atkin-Lehner level-reducing lemma, making use of the fact that $M_{t(\ell-1)}(SL_2(\mathbb{Z}))$ is spanned by the Eisenstein series (which are 1 mod 2) and the modular discriminant

$$\Delta(\tau) = q{f_1}^{24}.$$

First branch: Judge and Chen

With this many primes in hand, Zanello was able to give a proof of the entire families of implications for density:

Theorem

1. *If there exists an integer* $A \equiv \pm 1 \pmod 6$ *such that* $\delta_A > 0$, *and* δ_i *exists for all* $i \leq A$ *with* $i \equiv \pm 1 \pmod 6$, *then* $\delta_1 > 0$.
2. *If there exists an integer* $A \equiv 3 \pmod 6$ *such that* $\delta_A > 0$, *and* δ_i *exists for all* $i \leq A$ *with* $i \equiv 3 \pmod 6$, *then* $\delta_3 > 0$.

Second branch: m-regular partitions, m odd

About two years ago Zanello and I began looking at the m-regular partitions, which have generating function

$$\sum_{n=0}^{\infty} b_m(n)q^n = \frac{f_m}{f_1}.$$

Our first goal was to add to the body of literature on congruences and self-similarities of the forms

$$\sum_{n=0}^{\infty} b_m(An+B)q^n \equiv 0$$

$$\sum_{n=0}^{\infty} b_m(An+B)q^n \equiv \sum_{n=0}^{\infty} b_m(Cn+D)q^{jn}.$$

Second branch: m-regular partitions, m odd

We were able to produce quite a few. Here is a table of congruences from the paper, with some context from the literature:

m	$b_m(An+B)$ known to be even	Source
3		**New in this paper**
5	$b_5(2n)$, when n is not twice a pentagonal number	Calkin *et al.* [7]
7	Family modulo $2p^2$ if $(\frac{-14}{p}) = -1$, p prime	Baruah and Das [2]
9	$b_9(2^j n + c(j))$ for various j	Xia and Yao [36] **More new in this paper**
11	$b_{11}(22n+2, 8, 12, 14, 16)$ (finite family)	Zhao, Jin, and Yao [38]
13	$b_{13}(2n)$, when $n \neq k(k+1)$, $n \neq 13k(k+1)+3$	Calkin *et al.* [7]
15		**We conjecture none**
17	$b_{17}(2 \cdot 17^{2\alpha+2} p^{2\beta} n + c(p, \alpha, \beta))$ if $(\frac{-51}{p}) = -1$	Zhao, Jin, and Yao [38]
19	$b_{19}(38n+2, 8, 10, 20, 24, 28, 30, 32, 34)$ (finite family)	Radu and Sellers [31] **More new in this paper**
21		**New in this paper**
23	Family modulo $2p^2$ if $(\frac{-46}{p}) = -1$, p prime	Baruah and Das [2]
25	$b_{25}(100n+64, 84)$ (finite family)	Dai [10] **More new in this paper**
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Second branch: m-regular partitions, m odd

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A couple of observations stand out. The moduli 15 and 27 are interesting since for some reason there seem to be no congruences, unlike most other moduli. Notice the lack of many congruences in the literature for $m = 11$, and our inability to add any at the time.

Second branch: m-regular partitions, m odd

We were focusing on m odd because in our initial search for candidates, the even-regular partitions appeared to have either:

1. None - no congruences whatsoever; or
2. Many, being in fact lacunary mod 2.

For instance, $b_2(n)$ is odd if and only if n is a pentagonal number; meanwhile, the 6-regular partitions appear to lack even arithmetic progressions entirely.

Second branch: m-regular partitions, m odd

This is in line with what seems to be a very broad but so far empirically supported expansion of Parkin-Shanks:

Conjecture

Let $F(q) = \sum_{n \geq 0} c(n) q^n$ be an eta-quotient, and denote by δ_F the odd density of its coefficients $c(n)$. We have:
i) For any F, δ_F exists and satisfies $\delta_F \leq 1/2$.
ii) If $\delta_F = 1/2$, then $c(Am + B)$ has odd density $1/2$ for all arithmetic progressions $Am + B$.
iii) If $\delta_F < 1/2$, then the coefficients of F are identically zero (mod 2) on some arithmetic progression.
iv) If the coefficients of F are not identically zero (mod 2) on any arithmetic progression, then they have odd density $1/2$ on every arithmetic progression; in particular, $\delta_F = 1/2$.
(Note: i), ii), and iii) together imply iv), and that iv) implies iii).)

Second branch: m-regular partitions, m odd

Some sample theorems from that paper:

Theorem

It holds that $\sum_{n=0}^{\infty} b_{19}(10n+8)q^n \equiv q\sum_{n=0}^{\infty} b_{19}(2n)q^{5n}$. *By iteration,*

$$b_{19}\left(2\cdot 5^{2d}(50n+10k+8)+9((5^{2d}-1)/24)\right)\equiv 0,$$

for all $d, k \geq 1$ *with* $k \not\equiv 1 \pmod 5$.
If $p \equiv 13, 17, 19,$ *or* $23 \pmod{24}$ *is prime, then*

$$b_{21}(4(p^2n+kp-11\cdot 24^{-1})+1)\equiv 0$$

for all $1 \leq k < p$, *where* 24^{-1} *is taken modulo* p^2.

Second branch: m-regular partitions, m odd

Our proof techniques were either algebraic manipulations, using known dissections of eta-quotients, standard modular verifications of similarities, or Radu's machine.

Theorem such as the one for b_{21}, giving even arithmetic progressions for infinite set of primes, arise from the algebraic proofs. For instance, we could establish by dissection that

$$\sum_{n=0}^{\infty} b_{21}(4n+1)q^n \equiv \left(\frac{f_3^3}{f_1}\right) f_3$$

which means that the sequence $b_{21}(4n+1)$ mod 2 represents integers writeable as $m(3m-2) + 3(n/2)(3n-1)$, and analysis of quadratic residues yields avoided progressions.

Second branch: m-regular partitions, m odd

We also conjectured the following noncongruences:

Conjecture

Let $m \in \{6, 10, 14, 15, 18, 20, 22, 26, 27, 28\}$. *We have:*

1. *For no integers* $A > 0$ *and* $B \geq 0$, $b_m(An + B) \equiv 0$ *for all* n.
2. *The series* f_m/f_1 *has odd density* $1/2$.

The odd moduli 15 and 27 are certainly the most interesting here. With the other m-regular partitions, when m is odd, have so many even progressions, why don't these?

Grafting the branches: m-regular partitions, m even

Conjecture

Let $m \in \{6, 10, 14, 15, 18, 20, 22, 26, 27, 28\}$. We have:

1. *For no integers $A > 0$ and $B \geq 0$, $b_m(An + B) \equiv 0$ for all n.*
2. *The series f_m/f_1 has odd density $1/2$.*

The missing small evens here are, on the other hand, lacunary: $b_2(n)$ is only odd if n is a pentagonal number, $b_4(n)$ if n is triangular, and so on. The others have something more complex going on, which is the next and final phase of the talk.

Grafting the branches: m-regular partitions, m even

First let's dispose of the cases guaranteed to be lacunary.

It follows from a theorem of Gordon and Ono that $\frac{f_m}{f_1}$ will be lacunary mod 2 if $m = m_0 2^j$ with $2^j > m_0$. This covers any power of 2; 12, 24, 48, etc.; 40, 80, 160, etc. These should have lots of even progressions.

This means there is a restricted range of values for m that are potentially interesting for us.

Grafting the branches: m-regular partitions, m even

When we examined the actual placement of odd values in the even-regular partitions, we found something very interesting: within some progressions, they frequently appear to be congruent to odd-power multipartitions!

If our conjecture about t-multipartitions is true, then, none of these progressions will have any subprogressions identically 0 mod 2, and ought to have relative density $1/2$.

Grafting the branches: m-regular partitions, m even

Formally, among many others:

Theorem

For all $n \geq 0$, the following coefficients are of equal parity:

In $q\frac{f_{10}}{f_1}$ and $\frac{1}{f_5^3}$: q^{25n+5} and q^{25n+15}

In $q\frac{f_{20}}{f_1}$ and $\frac{1}{f_5}$: q^{25n+5} and q^{25n+10}.

In $q^5\frac{f_{22}}{f_1}$ and $\frac{1}{{f_{11}}^9}$: $q^{121n+11}$, $q^{121n+55}$, $q^{121n+66}$, $q^{121n+77}$, $q^{121n+99}$.

In $q^5\frac{f_{44}}{f_1}$ and $\frac{1}{{f_{11}}^7}$: $q^{121n+11}$, $q^{121n+22}$, $q^{121n+33}$, $q^{121n+77}$, $q^{121n+110}$

In $q^5\frac{f_{88}}{f_1}$ and $\frac{1}{{f_{11}}^3}$: $q^{121n+22}$, $q^{121n+33}$, $q^{121n+44}$, $q^{121n+66}$, $q^{121n+99}$.

Grafting the branches: m-regular partitions, m even

It turns out that the proof of these comes from our first paper! Each of those Ramanujan-Kolberg identities can be modified slightly to give congruences such as these.

Let's sketch the proof for the multiples of 11: the 22-, 44-, and 88-regular partitions. (Notice that this is our full interesting range: $2^4 > 11$, so further instances are lacunary.)

Grafting the branches: m-regular partitions, m even

We begin with the $(11, 6, 1)$ identity from our original Ramanujan-Kolberg theorem:

$$q \sum_{n=0}^{\infty} p(11n+6)q^n \equiv \frac{1}{{f_1}^{11}} + \frac{1}{f_{11}}.$$

Grafting the branches: m-regular partitions, m even

Recall that the operator $|U(m)$ acts on a power series by extracting the terms q^{mi} and substituting $q^{mi} \to q^i$. It has the property that

$$\left(\sum_{n=0}^{\infty} a(n)q^n\right)\left(\sum_{k=0}^{\infty} b(k)q^{mk}\right)|U(m) = \left(\sum_{n=0}^{\infty} a(n)q^n\right)|U(m)\left(\sum_{k=0}^{\infty} b(k)q^k\right).$$

Grafting the branches: m-regular partitions, m even

So our identity can be reworded

$$q^5 \frac{1}{f_1} | U(11) \equiv \frac{1}{{f_1}^{11}} + \frac{1}{f_{11}}.$$

Now multiplying through by powers of f_1, we get

$$q^5 \frac{f_{22}}{f_1} | U(11) \equiv \frac{1}{{f_1}^{9}} + \frac{f_2}{f_{11}}$$
$$q^5 \frac{f_{44}}{f_1} | U(11) \equiv \frac{1}{{f_1}^{7}} + \frac{f_4}{f_{11}}$$
$$q^5 \frac{f_{88}}{f_1} | U(11) \equiv \frac{1}{{f_1}^{3}} + \frac{f_8}{f_{11}}.$$

Grafting the branches: m-regular partitions, m even

$$q^5 \frac{f_{22}}{f_1} | U(11) \equiv \frac{1}{{f_1}^9} + \frac{f_2}{f_{11}}$$
$$q^5 \frac{f_{44}}{f_1} | U(11) \equiv \frac{1}{{f_1}^7} + \frac{f_4}{f_{11}}$$
$$q^5 \frac{f_{88}}{f_1} | U(11) \equiv \frac{1}{{f_1}^3} + \frac{f_8}{f_{11}}.$$

Now we simply observe that f_2, f_4, and f_8 are odd at respectively 2, 4, and 8 times the pentagonal numbers, which are

$$\{(n/2)(3n-1)\} \equiv \{3 \cdot 2^{-1} x^2 - 24^{-1}\} \pmod{11}$$

or, indeed, modulo any whole number coprime to 6.

Grafting the branches: m-regular partitions, m even

Now observe that $\{3 \cdot 2^{-1}x^2 - 24^{-1}\}$ avoids half the residue classes mod 11, determined by quadratic nonresidues, and multiplying by $1/f_{11}$ does not change this avoidance.

Hence in those progressions, the left-hand sides and the first term of the right-hand sides are congruent.

A quick tangent back: b_{11}

Briefly if I have time at this point: can we use this viewpoint to fix our previous lack and find some new congruences for b_{11}?

If we multiply through by a single f_1, we get:

$$q^5 \frac{f_{11}}{f_1} | U(11) \equiv \frac{1}{{f_1}^{10}} + \frac{f_1}{f_{11}}.$$

The first term on the left is just a doubling of the 5-multipartition function. It's never odd for q^{2n+1}. Okay, so what does f_1/f_{11} avoid? Mod 22, we observe that f_1 avoids $\{3, 6, 8, 9, 10, 14, 17, 19, 20, 21\}$.

A quick tangent back: b_{11}

Hence $q^5 \frac{f_{11}}{f_1} | U(11)$ avoids $22n + (3, 9, 17, 19, 21)$, and therefore

Theorem

It holds that $b_{11}(242n + B) \equiv 0 \pmod 2$ *for* $n \geq 0$ *and* $B \in \{28, 94, 182, 204, 226\}$.

Grafting the branches: m-regular partitions, m even

For all the smallest moduli m_0 where we have these identities - $\{5, 7, 11, 13, 17, 19, 23\}$ - some facts about the $m_0 \cdot 2^k$-regular partitions are thus equivalent to facts about multipartitions and, by the earlier work, imply facts about the density of odd $p(n)$.

While Chen's proof of Judges' conjecture shows us that identities exist for higher moduli, once the moduli start getting above 24, more complex behaviors arise.

Grafting the branches: m-regular partitions, m even

For instance, the three-term identity for $m = 25$ is

$$q^2 \sum_{n=0}^{\infty} p(25n+24)q^n \equiv \frac{1}{{f_1}^{25}} + \frac{1}{{f_5}^5} + \frac{q}{f_1}.$$

A corresponding behavior for even-regular partitions is

Theorem

For all $n \geq 0$, the following coefficients are of equal parity:
In $q\frac{f_{400}}{f_1}$ and $\frac{1}{f_{25}^9}$: $q^{1125n+75}$, $q^{1125n+225}$, $q^{1125n+450}$, $q^{1125n+975}$.

Grafting the branches: m-regular partitions, m even

When we multiply through the original identity, we find the following to analyze:

$$q^{26}\frac{f_{400}}{f_1}|U(25)+\frac{1}{{f_1}^9}\equiv\frac{f_{16}}{{f_5}^5}+q{f_1}^{15}.$$

And indeed, $\frac{f_{16}}{{f_5}^5}$ avoids progressions 3 or 4 mod 5, and coefficients in ${qf_1}^{15}$ are even outside of integers representable as $1+4\binom{n+1}{2}+\binom{m+1}{2}$, which misses 3, 9, 18, and 39 mod 45; this proves the theorem.

Grafting the branches: m-regular partitions, m even

These get more challenging as m_0 gets larger. For instance, establishing our results for b_{200} requires analyzing

$$q^{26}\frac{f_{200}}{f_1}|U(25) \equiv \frac{1}{{f_1}^{17}} + q{f_1}^7 + \frac{f_8}{{f_5}^5}.$$

For this we need to know the location of odd coefficients in $q{f_1}^7$, which was quite recently established by Cherubini and Mercuri.

Without exact information, however, we can still establish some "almost always" congruences.

Grafting the branches: m-regular partitions, m even

For an extreme case, take $m = 65537 \cdot 2^{16}$. For some c, we have:

$$q^c \frac{{f_{65537}}^{65536}}{f_1} | U(65537) \equiv \frac{1}{f_1} + (\text{lacunary series}) + \epsilon \frac{f_{65536}}{f_{65537}}.$$

The series in the middle are eta-products, and are lacunary mod 2 by a theorem of several students of Ono.

Thus, depending on ϵ, there is an arithmetic progression mod 65537 in $b_{65537 \cdot 2^{16}}(n)$ which, almost all the time or almost half the time, matches the parity of the partition function!

Open questions

1. Find a recipe for the ϵ in the Judge-Chen congruences. Every expression thereby expanded gives congruences for multipartition functions and regular partitions.
2. Sub-question of the above: find moduli where "many" of the ϵ are 0. These will give tidy identities.
3. Extend Chen's proof to non-prime-powers.
4. Explain why δ_1 and δ_3 are in separate families, or give a cross-implication between them.