

Schmidt-type theorems for partitions with uncounted parts

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Outline

- ① Main theorem; combinatorial proof
- ② q -series proofs of specific cases and their related identities
- ③ Where to go from here?

Introduction

In September of last year, George Andrews in this Seminar discussed and generalized Schmidt's theorem, which originally appeared as a problem in the *American Mathematical Monthly*:

Theorem

Let $p(n)$ be the number of partitions $\lambda_1 + \lambda_2 + \dots$ of the integer n , and let $f(n)$ denote the number of partitions $\pi_1 + \pi_2 + \dots$ into distinct parts $\pi_i > \pi_{i+1}$ such that $n = \pi_1 + \pi_3 + \pi_5 + \dots$. Then $f(n) = p(n)$ for all $n \geq 1$.

Introduction

It turns out that this theorem is a specific case of a quite general theorem on partitions in which some parts are counted and some not, which it is our goal to prove today.

Our method of proving the main theorem is bijective - a colored generalization of a partition map first invented by Dieter Stockhofe, from m -distinct to m -regular partitions.

The main theorem

The full theorem permits an enormous degree of freedom:

Theorem

Fix $m \geq 2$. Let $S = \{s_1, \dots, s_i\} \subseteq \{1, \dots, m-1\}$ with $1 \in S$, and $\vec{\rho} = (\rho_1, \dots, \rho_{m-1})$. Denote by $P_{m,S}(n; \vec{\rho})$ the number of partitions $\lambda = (\lambda_1, \dots, \lambda_c)$ into parts repeating less than m times in which

$$n = \sum_{c \equiv s_j \pmod{m}} \lambda_c, \text{ and}$$

$$\rho_k = \sum_{c \equiv k \pmod{m}} \lambda_k - \lambda_{k+1}.$$

Then $P_{m,S}(n; \vec{\rho})$ is also the number of partitions of n where parts $k \pmod i$ have $s_{k+1} - s_k$ colors (set $s_{i+1} = m$), and, labeling colors of parts $k \pmod i$ by s_k through $s_{k+1} - 1$, the color j appears ρ_j times.

Corollaries

To thin the forest of parameters out a bit, here are some corollaries that perhaps more obviously exhibit the fact that this generalizes Schmidt.

Set $m = 2$ and $S = \{1\}$. Let $p(n, \ell)$ be the number of partitions of n with exactly ℓ parts. Then the theorem becomes

Corollary

Let $f(n, \ell)$ denote the number of partitions $\pi_1 + \pi_2 + \dots$ into distinct parts $\pi_i > \pi_{i+1}$ such that $n = \pi_1 + \pi_3 + \pi_5 + \dots$ and $\ell = \pi_1 - \pi_2 + \pi_3 - \pi_4 + \dots$. Then $f(n, \ell) = p(n, \ell)$ for all $n \geq 1$.

Summing over all ℓ we get the original Schmidt theorem.

Corollaries

Another summed version:

Corollary

Fix $m > 2$ and $1 \leq i < m$. Let $R_{m,i}(n)$ be the set of partitions $\lambda_1 + \lambda_2 + \dots$, $\lambda_1 \geq \lambda_2 \geq \dots > 0$, in which parts can only appear fewer than m times, and in which $n = \sum_{s=0}^{\infty} \sum_{j=1}^i \lambda_{sm+i}$. Then

$$\sum_{n=0}^{\infty} R_{m,i}(n)q^n = \frac{1}{(q; q)_{\infty} (q^i; q^i)_{\infty}^{m-i-1}}.$$

Here we have set $S = \{1, 2, \dots, i\} \subseteq \{1, 2, \dots, m-1\}$. We end up getting one color of everything not a multiple of i , and $m-i$ colors of multiples of i .

Corollaries

The first nontrivial case not covered by the previous corollaries is the following.

Corollary

The number of partitions in which parts repeat less than 4 times and

$$n = \lambda_1 + \lambda_3 + \lambda_5 + \dots$$

is equal to the number of partitions in which even parts appear in one color and odd parts appear in two colors.

One observes that the latter set is well-known to have the same generating function as the much-studied *overpartitions*.

Background

There are three classes of partitions known to be equinumerous:

- ① those with parts not divisible by m , the *m-regular* partitions;
- ② those into parts appearing less than m times, the *m-distinct* partitions;
- ③ those into parts differing by less than m (and with first part less than m), the *m-flat* partitions.

The first two are the subject of many classical maps by Sylvester, Glaisher, Franklin, etc., each with various useful combinatorial properties. The last two are obviously conjugates of each other.

The first and third have a much less widely known direct map ϕ produced by Dieter Stockhofe in his 1982 Ph.D. thesis, which will be crucial to our argument today.

The map ϕ

We define two vector operations on partitions:

$$n\lambda = (n\lambda_1, n\lambda_2, \dots)$$

$$\lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots).$$

Here we consider a partition to include an infinite sequence of trailing zeroes (also convenient for simplifying the definition of m -flat partitions).

The map ϕ

The following fact is easy to prove:

Lemma

Let $\vec{v} = (v_1, \dots, v_k)$ be a sequence of nonzero residues modulo m . Then there is a unique partition $\lambda(\vec{v})$ which is m -regular, m -flat, and for which $\lambda(\vec{v})_i \equiv v_i \pmod{m}$.

Example

If $m = 5$ and $\vec{v} = (1, 1, 2, 1, 4, 3)$, then $\lambda(\vec{v}) = (11, 11, 7, 6, 4, 3)$.

The map ϕ

From this, we see that any m -regular partition λ can be written as

$$\lambda(\vec{v}) + m\mu,$$

where \vec{v} is the residue vector of λ itself, and μ is a partition with number of parts not more than the number of parts of λ .

Indeed the set of m -regular partitions is just exactly the union of all such sums over all residue vectors \vec{v} and μ of valid length.

The map ϕ

A good way to see this is the *m-modular diagram*, which is perhaps a little less commonly known than the Ferrers diagram of a partition, so let us define it here.

If $\lambda_i = km + j$, $0 \leq j < m$, then write in the i -th row k copies of m , followed by j if nonzero.

Example

If $\lambda = (18, 16, 8, 7, 6, 3)$, then the 5-modular diagram of λ is

| | | | |
|---|---|---|---|
| 5 | 5 | 5 | 3 |
| 5 | 5 | 5 | 1 |
| 5 | 3 | | |
| 5 | 2 | | |
| 5 | 1 | | |
| 3 | | | |

The map ϕ

Stockhofe's map is best understood as moving units of m around in the diagram. We can see where the "excess" units are located for m -regular partitions; what we want is a way to uniquely add these parts to $\lambda(\vec{v})$ in such a way that the resulting partition remains flat.

Because it's more useful for us today, and because I think it's a little easier to present, I will do this in the reverse direction. We'll also need to add consideration of color later.

The map ϕ

Let λ be an m -flat partition. Initialize $\mu = ()$, the empty partition.

Working from the smallest to the largest part, remove from λ any parts divisible by m for which, after removal, the partition is still m -flat. Append these parts to μ . These will be parts such that

- $\lambda_i = km = \lambda_{i-1}$, i.e. all but the first of a repeated part divisible by m ;
- $\lambda_1 = km$, i.e. the largest part is divisible by m ; OR
- parts $\lambda_i = k_i m$, $i > 1$, such that $\lambda_{i-1} = k_i m + j_1$, $\lambda_{i+1} = (k_i - 1)m + j_2$, with $0 < j_1 < j_2 < m$.

The latter, in other words, are multiples of m between two nonmultiples of m that differ by less than m .

Call the remaining partition λ^- .

The map ϕ

The remaining parts divisible by m in λ^- are all distinct, not the largest (or smallest) parts, and any remaining part $\lambda_i = k_i m$ lies between $\lambda_{i-1} = k_i m + j_1$ and $\lambda_{i+1} = (k_i - 1)m + j_2$ with $0 < j_2 \leq j_1 < m$. Hence it is possible to leave an m -flat partition by removing λ_i and also subtract m from every larger part.

The map ϕ

Working from the largest to the smallest remaining part, for each remaining part $\lambda_i = k_i m$, remove λ_i and subtract m from all parts λ_j , $j < i$. Append $\lambda_i + (i - 1)m$ to μ as a part.

After all of these parts are removed, the remaining partition is m -flat and m -regular, and hence is automatically $\lambda(\vec{v})$ for the residue vector of the original partition. Now construct

$$\phi(\lambda) = \lambda(\vec{v}) + m\mu'.$$

Example

Let $m = 5$, $\lambda = (22, 19, 15, 15, 13, 10, 6, 5, 2)$. This is 5-flat but not 5-regular. Write its 5-modular diagram:

| | | | | |
|---|---|---|---|---|
| 5 | 5 | 5 | 5 | 2 |
| 5 | 5 | 5 | 4 | |
| 5 | 5 | 5 | | |
| 5 | 5 | 5 | | |
| 5 | 5 | 3 | | |
| 5 | 5 | | | |
| 5 | 1 | | | |
| 5 | | | | |
| 2 | | | | |

Example

We have $\mu = (3, 1)$ so far, and $\lambda^- = (22, 19, 15, 13, 10, 6, 2)$, with 5-modular diagram:

| | | | | |
|---|---|---|---|---|
| 5 | 5 | 5 | 5 | 2 |
| 5 | 5 | 5 | 4 | |
| 5 | 5 | 5 | | |
| 5 | 5 | 3 | | |
| 5 | 5 | | | |
| 5 | 1 | | | |
| 2 | | | | |

We now first remove the red units, and then the blue.

Example

We end up with $\lambda(\vec{v}) = (12, 9, 8, 6, 2)$, and $\mu = (5, 5, 3, 1)$.
Conjugate to get $5\mu' = 5 \cdot (4, 3, 3, 2, 2)$ and add:

| | | | | | | |
|---|---|---|---|---|---|---|
| 5 | 5 | 5 | 5 | 5 | 5 | 2 |
| 5 | 5 | 5 | 5 | 4 | | |
| 5 | 5 | 5 | 5 | 3 | | |
| 5 | 5 | 5 | 1 | | | |
| 5 | 5 | 2 | | | | |

We get $\phi(\lambda) = (32, 29, 23, 16, 12)$.

Proof of the main theorem

Theorem

Fix $m \geq 2$. Let $S = \{s_1, \dots, s_i\} \subseteq \{1, \dots, m-1\}$ with $1 \in S$, and $\vec{\rho} = (\rho_1, \dots, \rho_{m-1})$. Denote by $P_{m,S}(n; \vec{\rho})$ the number of partitions $\lambda = (\lambda_1, \dots)$ into parts repeating less than m times in which

$$n = \sum_{c \equiv s_j \pmod{m}} \lambda_c, \text{ and}$$

$$\rho_k = \sum_{c \equiv k \pmod{m}} \lambda_k - \lambda_{k+1}.$$

Then $P_{m,S}(n; \vec{\rho})$ is also the number of partitions of n where parts $k \pmod{m}$ have $s_{k+1} - s_k$ colors (set $s_{i+1} = m$), and, labeling colors of parts $k \pmod{m}$ by s_k through $s_{k+1} - 1$, the color j appears ρ_j times.

Proof of the main theorem

Fix m and the set S of places to be counted. View the partition in conjugate.

Now m -distinctness becomes m -flatness. Each part will intersect some counted and some uncounted parts, altering the weight it contributes.

A given weight can appear in multiple ways if multiple uncounted parts can appear as the final nodes in its row; this gives rise to the color naming we instituted, assigning to each part the color of the residue of its natural size mod m .

Proof of the main theorem

The number of colors available for a part of size $j \pmod i$ is exactly 1 more than the number of uncounted boxes that may arise after $j \pmod i$ are filled, which is in turn $s_{j+1} - s_j$, with the exception of parts that are multiples of m , which will be removed by ϕ .

Proof of the main theorem

When we apply ϕ , we now must take account of color. The only parts being moved are the parts that are multiple of m , contributing parts we would call $(mk)_m$. These are the color we want to go away.

We institute the convention that adding multiples of parts with color 5 leaves the new part with the color of the other summand.

Proof of the main theorem

The vector counting the differences by place residue mod m becomes exactly the count of parts of each color appearing, and this is fixed by ϕ with our color convention except for color m . The map takes an m -flat λ to an m -regular $\phi(\lambda)$, so there are no parts of color m remaining.

The total reduced weight does not change as multiples of m are removed from any one part and added to others, m at a time. Thus the properties of the map immediately imply the equal cardinality of the two sets. □

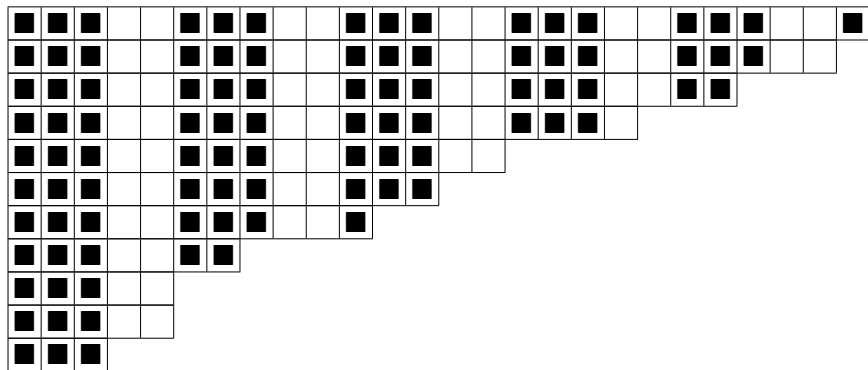
Example of the main theorem

For example, let $m = 5$, $S = \{1, 2, 3\}$, and

$\lambda = (11, 11, 11, 10, 10, 8, 8, 7, 7, 7, 7, 6, 6, 5, 5, 4, 4, 4, 4, 3, 3, 3, 2, 2, 2, 1)$.

Example of the main theorem

In conjugate, we have $\lambda' = (26, 25, 22, 19, 15, 13, 11, 7, 5, 5, 3)$, with a complex coloration. Denoting counted and uncounted parts, we have this diagram:



Example of the main theorem

If we count the weights contributed by each part of λ' , we have

$$|\lambda'| = 16 + 15 + 14 + 12 + 9 + 9 + 7 + 5 + 3 + 3 + 3 = 96.$$

There is one color each of part sizes 1 or 2 mod 3, and three colors of multiples of 3.

Counting the reduced weight of the partition given by the specified parts and calculating the differences among part sizes in places mod 5 in the original λ , we find that λ is counted in

$$P_{5,\{1,2,3\}}(96, (2, 2, 2, 1)).$$

Example of the main theorem

We apply ϕ and obtain

$$\phi(\lambda) = (41, 27, 24, 23, 21, 12, 3).$$

We will next reduce the weight by the counted parts, 3 for every 5. With our color convention, we color parts 1 or 2 mod 3 with subscript 1 and 2 respectively, and multiples of 3 with subscript 3 or 4 according as the original part was 3 or 4 mod 5. We obtain

$$(25_1, 17_2, 15_4, 15_3, 13_1, 8_2, 3_3).$$

Example of the main theorem

Final partition:

$$(25_1, 17_2, 15_4, 15_3, 13_1, 8_2, 3_3).$$

Recall that λ was originally in

$$P_{5, \{1,2,3\}}(96, (2, 2, 2, 1)).$$

We see here that this matches the color count with two each of colors 1, 2, and 3, and one part of color 4.

Remarks and variations

Remark 1: The bijection preserves not only the total counts but the ordered residue-vector, which is even more precise than the theorem as stated; however, that would have been an even more elaborate statement.

Remark 2: The refinement of the original Schmidt by counts can actually be obtained by analyzing Mork's proof, which positions hooks on the main diagonal of a partition.

Remarks and variations

Remark 3: In their Partition Analysis paper on Schmidt's Theorem, Andrews and Paule also proved that, if one considers the set of arbitrary partitions with no restrictions on parts and only adds parts in odd places, one obtains two-colored partitions.

This is even easier to prove from colored-conjugate viewpoint: simply observe that in the conjugate, arbitrary partitions are possible and each part size can appear in two different colors. The places of parts to be counted can be arbitrarily generalized and the resulting part sizes listed, additional colors in a part size occurring for each uncounted place.

q -series proofs

Now we'd like to do something a little unusual, which is to prove some specific cases of the main theorem with q -series techniques.

There is interest here since the sum-product identities that arise seem potentially useful or worth considering in their own right.

There's also the challenge – the q -series side of this proof seems considerably harder!

q -series proofs

We'll need some tools:

The q -hypergeometric series is defined by

$${}_r\phi_s \left[\begin{matrix} a_1, & a_2, & \dots, & a_r \\ b_1, & \dots, & b_s \end{matrix} ; q, z \right] \\ = \sum_{n=0}^{\infty} \frac{(a_1; q)_n \dots (a_r; q)_n}{(b_1; q)_n \dots (b_s; q)_n} \left((-1)^n q^{\binom{n}{2}} \right)^{1+r-s} z^n.$$

q -series proofs

The q -Chu-Vandermonde summation is

$${}_2\phi_1 \left[\begin{matrix} q^{-n}, & b \\ c \end{matrix}; q, cq^n/b \right] = \frac{(c/b; q)_n}{(c; q)_n}$$

or equivalently

$${}_2\phi_1 \left(\begin{matrix} q^{-n} & b \\ c \end{matrix}; q; q \right) = \frac{b^n (c/b; q)_n}{(c; q)_n}$$

and the q -binomial theorem is

$$\sum_{n=0}^{\infty} z^n q^{\binom{n+1}{2}} \begin{bmatrix} N \\ n \end{bmatrix}_q = (-zq; q)_N.$$

We'll begin with the $m = 3$, $S = \{1, 2\}$ case: partitions into parts repeating no more than twice with parts counted only in places not divisible by 3, are in bijection with ordinary partitions into exactly m parts when in the original partition

$$m = \lambda_1 - \lambda_3 + \lambda_4 - \lambda_6 + \lambda_7 - \lambda_9 + \dots$$

q -series proofs

Strategy:

- 1 Set up a recurrence.
- 2 Solve to obtain a sum expression.
- 3 Show that this sum is also the desired product.

Let

$$P_N = P_N(x_1, x_2, x_3, \dots, x_N; t)$$

be the generating function for partitions with exactly N parts but zeros allowed, of the form

$$\lambda_1 + \lambda_2 + \dots + \lambda_N \quad (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_N \geq 0)$$

where no part (including 0) appears more than twice, the exponent of x_i is λ_i , and the exponent of t is

$$\lambda_1 - \lambda_3 + \lambda_4 - \lambda_6 + \lambda_7 - \lambda_9 + \dots$$

Denote

$$X_i = x_1 x_2 \dots x_i t^{\chi_3(i)},$$

where $\chi_3(i) = 0$ if $3|i$ and 1 otherwise.

A little combinatorial thought gives that the P_N satisfy the recurrence

$$P_N = \frac{X_{N-2} P_{N-2}}{1 - X_N} + \frac{X_{N-1} P_{N-1}}{1 - X_N}, \quad (1)$$

with $P_0 = 1$, $P_1 = \frac{1}{1-X_1}$, and $P_2 = \frac{1}{(1-X_1)(1-X_2)}$. This recursion and the initial conditions completely define P_N .

If we gather terms and write

$$P_N = \frac{\pi_N}{\prod_{i=1}^N (1 - X_i)},$$

then the π_N are defined by the initial conditions and recursion
 $\pi_0 = \pi_1 = \pi_2 = 1$ and for $N > 2$,

$$\pi_N = (1 - X_{N-1})X_{N-2}\pi_{N-2} + X_{N-1}\pi_{N-1}. \quad (2)$$

We now make the substitution

$$x_i = \begin{cases} q & \text{if } 3 \nmid i \\ 1 & \text{if } 3 \mid i. \end{cases}$$

This makes

$$X_i = q^{\lceil \frac{2i}{3} \rceil} t^{\chi_3(i)}.$$

In doing so, we see that the function $P_N(q, q, 1, q, q, 1, \dots; t)$ has the coefficient of $q^n t^m$ counting the number of partitions into exactly N nonnegative parts, in which any part size including 0 must appear less than 3 times, with $n = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \dots$, and in which $m = \lambda_1 - \lambda_3 + \lambda_4 - \lambda_7 + \dots$.

q -series proofs

The summation of the recurrence gives:

Lemma

For $n \geq 0$ and $r \in \{0, 1, 2\}$, with the above substitutions for the X_i ,

$$\begin{aligned} \pi_{3n+r}(q, q, 1, \dots; t) \\ = \sum_{j=0}^n (-1)^j q^{n^2 - n + r(n+j) + \binom{j+n+1}{2}} (-q; q)_{n-j} \begin{bmatrix} n \\ j \end{bmatrix}_{q^2} t^{j+n}. \end{aligned}$$

Proof.

Show that the formulas satisfy the recursion for each residue of N mod 3. In each case, compare coefficients of t^k on both sides. \square

q -series proofs

Note that we only need N multiples of 3, since this will count partitions of N , $N - 1$, and $N - 2$, so we sum these.

We find that we need to show:

$$\sum_{n \geq j \geq 0} \frac{(-1)^j q^{n^2 - n + \binom{j+n+1}{2}} (-q; q)_{n-j} \begin{bmatrix} n \\ j \end{bmatrix}_{q^2} t^{j+n}}{(tq; q)_{2n} (q^2; q^2)_n} = \frac{1}{(tq; q)_\infty}.$$

q -series proofs

After some substitutions to shift index, multiplying through by $(tq; q)_\infty$, and expanding some of the products, we find that we want to show the following triple indexed sum:

$$\sum_{m,n,j \geq 0} \frac{(-1)^{j+m} q^{(n+j)^2 - (n+j) + \binom{2j+n+1}{2} + \binom{m+1}{2} + m(2n+2j)}{(q^2; q^2)_j (q; q)_n (q; q)_m} t^{m+n+2j} = 1.$$

The coefficient of t^0 is 1, so what we finally want to show is that the coefficient of t^c for c positive is 0.

q -series proofs

We find that the coefficient of t^N is

$$\begin{aligned} & (-1)^N q^{\binom{N+1}{2}} \sum_{n,j \geq 0} \frac{(-1)^{n+j} q^{Nn-n+j^2-j}}{(q^2; q^2)_j (q; q)_n (q; q)_{N-n-2j}} \\ &= (-1)^N q^{\binom{N+1}{2}} \sum_{n \geq 0} \frac{(-1)^n q^{Nn-n}}{(q; q)_n (q; q)_{N-n}} \\ &\times \lim_{\tau \rightarrow 0} {}_2\phi_1 \left(\begin{matrix} q^{-N+n}, q^{-N+n+1}; q^2, q^{2(N-n)} \tau^{-1} \\ q/\tau \end{matrix} \right) \\ &= (-1)^N q^{\binom{N+1}{2}} \sum_{n \geq 0} \frac{(-1)^n q^{Nn-n + \binom{N-n}{2}}}{(q; q)_n (q; q)_{N-n}} \\ &= \frac{(-1)^N q^{N^2}}{(q; q)_N} \sum_{n \geq 0} (-1)^n q^{\binom{n}{2}} \begin{bmatrix} N \\ n \end{bmatrix}_q = \frac{(-1)^N q^{N^2}}{(q; q)_N} (1; q)_N = 0. \square \end{aligned}$$

Further questions

- 1 Can we use this map and/or the concept of uncounted parts to prove other identities concerning colored partitions?
- 2 How else can the concepts be varied?
- 3 Mork's original proof of Schmidt placed hooks. Hook-type identities are of much interest; can we create a hook-based combinatorial proof of these theorems?
- 4 Can the q -series proofs be generalized to the full theorem? What identity results?