Parity of the coefficients of certain eta-quotients, III: The case of pure eta-powers

Seminar in Partition Theory, q-Series, and Related Topics
(Rescheduled from AMS Southeastern Sectional)

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The goal

Today we study a very specific set of eta-products, pure eta-powers of the form $\prod_{k=1}^{\infty}(1-q^k)^t = \sum_{n=0}^{\infty} c_t(n)q^n$. Our results are twofold:

1. Given t, find an infinite set of non-nested arithmetic progressions $An+B$ on which $c_t(An+B)$ is even.
2. Given p prime, find an infinite class of t for which arithmetic progressions exist modulo p^2.

In the first you're only dealing with one product, so the second result might be called more substantial.

I'll now take a few slides to explain why we're interested in this question.

The parity of $p(n)$

The parity of the partition numbers is one of the big open questions in partition theory. If we define the *arithmetic density* of a property T of an integer-indexed sequence $\{f(n)\}_{n=n_0}^{\infty}$ to be

$$\delta(T) = \lim_{n\to\infty} \#\frac{1}{n}\{k < n | T \text{ holds for } f(k)\},$$

then the *Parkin-Shanks Conjecture* or, colloquially, the *Fifty-Fifty Conjecture* is that

Conjecture

The densities $\delta(p(n) \equiv 0 \pmod 2)$ and $\delta(p(n) \equiv 1 \pmod 2)$ both exist and equal 1/2.

The parity of $p(n)$

We are very far from settling this conjecture. The best bounds for the cardinalities of the even and odd partition numbers are from work of Bellaïche and Nicolas:

Theorem

$\#\{n < x | p(n) \textit{ is even }\} > c\sqrt{x}\ln\ln x$ *and*
$\#\{n < x | p(n) \textit{ is odd }\} > c\frac{\sqrt{x}}{(\ln x)^{7/8}}$.

So we do not even know if the densities, should they exist, are nonzero.

The parity of $p(n)$

Thanks to work of Cristian-Silviu Radu, completing work by Subbarao, Ono and others, we know:

Theorem

(Radu) There exists no nonzero integer pair (A, B) such that $p(An + B) \equiv 0 \pmod{2}$ for all $n \geq 0$.

The parity of $p(n)$

And yet closely related functions often do feature such progressions. For example, denote

$$f_i = \prod_{k=1}^{\infty}(1 - q^{ki}).$$

The generating function of partitions is $\frac{1}{f_1}$, those of the m-regular partitions are $\frac{f_m}{f_1}$, and those of the t-multipartitions are f_1^{-t}.

Importantly, we can relate the parity of $p(n)$ and these other functions. In this series of papers, Zanello, his student Judge, and I have been producing families of congruences, as well as relations and implications among these families.

The parity of p(n)

One such identity is this:

$$q\sum_{n=0}^{\infty} p(5n+4)q^n \equiv \frac{1}{{f_1}^5} + \frac{1}{f_5}.$$

Call δ_t the density, assuming it exists, of $\frac{1}{f_1^t} = \sum_{n=0}^{\infty} p_t(n)q^n$.

Corollary

If $\delta_1 = 1$, then δ_5 exists and is equal to $4/5$, with density zero for the odd coefficients of the series $\sum_{n=0}^{\infty} p_5(5n)q^{5n}$ and density 1 among all other coefficients.

Corollary

If the odd density of $p_5(n)$ is not distributed in this convoluted way, then $\delta_1 < 1$, and thus the even coefficients of $p(n)$ have positive density.

Context: previous work

In work on the regular partitions $\frac{f_m}{f_1} = \sum_{n=0}^{\infty} b_m(n)q^n$, Zanello and I found a number of congruences satisfied by the m-regular partitions. Frequently these were one-offs, isolated instances; however, we could sometimes find infinite families. An example would be for the 21-regular partitions, for which we found

Theorem

If $p \equiv 19, 37, 47, 65, 85, 109, 113, 115, 137, 139, 143,$ *or* $167 \pmod{168}$ *is prime, then* $b_{21}(4(p^2 n + kp - 5 \cdot 24^{-1}))$ *is even for all* $1 \leq k < p$, *where* 24^{-1} *is taken modulo* p^2.

Context: previous work

This family is a consequence of the dissection

$$\sum_{n=0}^{\infty} b_{21}(4n)q^n \equiv f_1^2 f_3 + q\left(\frac{f_3^3}{f_1}\right) f_{21}.$$

Putting aside the second term at the moment (f_3^3/f_1 is odd only at a quadratic sequence, but this is not obvious from simple arguments), the first is an *eta-product*, in which all the powers of f_i are positive.

It is already known that these are all lacunary modulo 2 (i.e., have odd density 0) by work of Cotron, Michaelsen, Stamm, and Zhu (hereinafter CMSZ):

Context: previous work

Theorem (CMSZ, Theorem 1.1)

Suppose $u, w \geq 0$. Let $F(q) = \frac{\prod_{i=1}^{u} f_{\alpha_i}^{r_i}}{\prod_{i=1}^{w} f_{\gamma_i}^{s_i}}$, and assume that

$$\sum_{i=1}^{u} \frac{r_i}{\alpha_i} \geq \sum_{i=1}^{w} s_i \gamma_i.$$

Then the coefficients of F are lacunary modulo 2.

Now, for general power series it is absolutely not the case that lacunarity mod 2 implies the existence of even arithmetic progressions. However, a motivating master conjecture of this series of papers is that for eta-quotients, it does.

Context: previous work

One thing it was difficult for us to do in previous papers was find these infinite families. We could occasionally find them for specific b_m. What we could not do was find results for a class of m.

In today's work, we study an even more specific subset of eta-products, pure eta-powers of the form $f_1^t = \sum_{n=0}^{\infty} c_t(n)q^n$. Our goals are twofold:

1. Given t, find an infinite set of non-nested arithmetic progressions $An + B$ on which $c_t(An + B)$ is even.
2. Given p prime, find an infinite class of t for which arithmetic progressions exist modulo p^2.

Theorems

Including these for completeness and useful ingredients:

Theorem

For m coprime to 6, we have $c_1(mn+B) \equiv 0$ *whenever* $2 \cdot 3^{-1}B + 36^{-1}$ *is not a quadratic residue modulo m.*

Theorem

For m coprime to 6, we have $c_3(mn+B) \equiv 0$ *whenever* $2B + 4^{-1}$ *is not a quadratic residue modulo m.*

These are because

$$f_1 \equiv \sum_{n\in\mathbb{Z}} q^{\frac{n}{2}(3n-1)} \quad \text{and} \quad f_1^3 \equiv \sum_{n=0}^{\infty} q^{\binom{n+1}{2}}.$$

So, for instance, for m coprime to 6 we may write

$$\frac{k}{2}(3k-1) \equiv 2^{-1}\cdot 3(k-6^{-1})^2 - 24^{-1} \pmod{m},$$

and so if

$$B \equiv 2^{-1}\cdot 3(k-6^{-1})^2 - 24^{-1} \pmod{m}$$
$$2\cdot 3^{-1}B + 36^{-1} \equiv (k-6^{-1})^2 \pmod{m}.$$

Hence in the progression $mn+B$, if $2\cdot 3^{-1}B+36^{-1}$ is not a quadratic residue mod m, the arithmetic progression will be even in $c_1(mn+B)$. $\square$

Theorems

Definition

We say that f_1^t is p^2-*even* at a prime p with base $r \in \{0, \ldots, p^2 - 1\}$ if it is the case that $c_t(p^2 n + kp + r) \equiv 0$ for all $k \in \{1, \ldots, p - 1\}$.

We make the following conjectures:

Conjecture

For any given $t \geq 1$ odd, f_1^t is p^2-even for a positive proportion of primes p for some base r depending on t and p.

Conjecture

For any given prime p, there exist infinitely many t such that f_1^t is p^2-even for some base r depending on t and p.

Theorems

The first conjecture is true for t a sum of two quadratic terms magnified by powers of 2:

Theorem

Let $t = a + b \cdot 2^e$, $a, b \in \{1, 3\}$, $e > 0$. Then f_1^t is p^2-even at p for some base r, when -2^e is a quadratic nonresidue modulo p if $a = b$, and when $-3 \cdot 2^e$ is a quadratic nonresidue modulo p if $a \neq b$. In particular, f_1^t is p^2-even for a set of relative density at least $1/2$ in the primes, if this density exists.

We can establish that the second conjecture is true for all primes other than 2 or those congruent to 1 mod 24:

Theorems

Theorem

We have that f_1^t *is* p^2*-even in the following cases for* p *prime,* r *taken mod* p^2*:*

- $t = 2^d + 3$: $p \equiv 23 \pmod{24}$, $r \equiv -(2^{d-3}3^{-1} + 2^{-3})$.
- *If* d *is even above we may take* $p \equiv 5 \pmod{6}$. *If* d *is odd then we may instead take* $p \equiv 13 \pmod{24}$.
- $t = 2^d + 1$: $p \equiv 7 \pmod{8}$, $r \equiv -3(2^{d-3} + 2^{-3})$.
- *If* d *is even above we may take* $p \equiv 3 \pmod{4}$, $p \geq 7$.
- $t = 3 \cdot 2^d + 1$: $p \equiv 23 \pmod{24}$, $r \equiv -(2^{d-3} + 2^{-3} \cdot 3^{-1})$.
- *If* d *is even above we may take* $p \equiv 5 \pmod{6}$. *If* d *is odd then we may take* $p \equiv 13 \pmod{24}$.
- *If* $t = 3 \cdot 2^d + 3$: $p \equiv 7 \pmod{8}$, $r \equiv -(2^{d-3} + 2^{-3})$.
- *If* d *is even above we may take* $p \equiv 3 \pmod{4}$.

Theorems

The clauses cover all odd primes other than those that are congruent to 1 mod 24. Thus, we have the following corollary.

Corollary

Given a fixed prime $p \geq 5$, $p \not\equiv 1 \pmod{24}$, *there exist infinitely many* $t = a + b \cdot 2^e$, $a, b \in \{1, 3\}$, $e > 0$, *for which* f_1^t *is* p^2*-even at* p *for some base* r.

An example of this is

Corollary

For $p = 5$, $t = 4^d + 3$, $d \geq 1$, *we have* $r \equiv t \pmod{25}$. *For* $p = 3$, $t = 3 \cdot 4^d + 3$, $d \geq 1$, *we have* $r \equiv t/3 \pmod{9}$ *(and the latter will always be 2 mod 3).*

For instance, $c_{15}(9n + 2)$ and $c_{15}(9n + 8)$ are always even.

Proof sketches

The proofs are simply a matter of representability of integers by quadratic forms.

For the first theorem, fix $t = a + b \cdot 2^e$, $a, b \in \{1, 3\}$. These are exactly those odd t for which we may write

$$f_1^t \equiv f_{1 \text{ or } 3} \cdot f_{1 \text{ or } 3}^{2^e}.$$

Thus t can only be odd if it is representable as the sum of two quadratics, one either the pentagonal or the triangular numbers, and the other (independently) the 2^e-magnified pentagonal or triangular numbers.

Proof sketches

Suppose $a = b = 1$. Then we have two pentagonal progressions, one magnified, and the terms N appearing with nonzero coefficient in their product must satisfy, for some $k_1, k_2 \in \mathbb{Z}$,

$$2^{-1} \cdot 3(k_1 - 6^{-1})^2 - 24^{-1} + (2^e)\left(2^{-1} \cdot 3(k_2 - 6^{-1})^2 - 24^{-1}\right) \equiv N \pmod{p^2}.$$

After we complete some squares and simplify some notation, we find that there must be some nonzero x and y such that

$$\left(\frac{x}{y}\right)^2 \equiv -2^e \pmod{p}.$$

Hence if -2^e is not a quadratic residue modulo p, then the chosen arithmetic progression cannot have nonzero coefficients mod 2 in f_1^t. The r is $-24^{-1} - 2^e 24^{-1}$. $\square$

Proof sketches

For the second theorem, fix p, a, and b and let e vary in $a + b \cdot 2^e$.

- If $p \equiv 3, 5 \pmod 8$, then 2 is a quadratic nonresidue modulo p, and so among -2^e and $-3 \cdot 2^e$, half the values will be quadratic nonresidues modulo p and the hypotheses of the previous theorem will be satisfied.
- If $p \equiv 7 \pmod 8$ then -1 is a quadratic nonresidue mod p and 2 is a quadratic residue mod p, so -2^e is always a quadratic nonresidue modulo p.
- If $p \equiv 1 \pmod 8$, then -1 and 2 are quadratic residues mod p, but if $p \equiv 17 \pmod{24}$ then 3 is a quadratic nonresidue and so $-3 \cdot 2^e$ is always a quadratic nonresidue modulo p.

The case that does not satisfy any of the above is $p \equiv 1 \pmod{24}$, as stated. $\square$

Further Theorems

Things rapidly get a lot harder after these base cases. The first case of a t with three separated 1s in its binary expansion is 21:

Theorem

We have that $c_{21}(49n + k) \equiv 0 \pmod{2}$ *for* $k \in \{14, 28, 35\}$.

But to prove this we needed to use (standard) modular form machinery.

Further Theorems

Here is an exponential nonexistence theorem:

Theorem

There exists no progression $An + B$ for which $c_{2^d-1}(An + B) \equiv 0$ for all d.

This follows from the fact that

$$\sum_{n=0}^{\infty} c_{2^d-1}(n)q^n = f_1^{2^d-1} = \frac{f_1^{2^d}}{f_1} \equiv \frac{f_{2^d}}{f_1},$$

which matches the partition function for its first 2^d coefficients.

Open questions

- Finish the case of $p \equiv 1 \pmod{24}$. Is it even true? Certainly it is for some, e.g. f_1^5 seems to be 73^2-even with base 1110.
- Numerical computation certainly suggests many additional progressions exist beyond these arguments. A notable example is f_1^{13}, which also seems to be p^2-even for primes 1 mod 6 in addition to the 5 mod 6 proved above.
- A theorem of Chen gives arithmetic progressions for any c_{3k}, but the moduli have many prime factors. Ours apply to fewer t but are mod p^2. Can we be even more parsimonious and find classes of arithmetic progressions $pn + B$ for primes p?

Thank you!

Master Conjecture

Conjecture

Let $F(q) = \sum_{n \geq 0} c(n) q^n$ be an eta-quotient, and denote by δ_F the odd density of its coefficients $c(n)$. We have:
i) For any F, δ_F exists and satisfies $\delta_F \leq 1/2$.
ii) If $\delta_F = 1/2$, then $c(Am + B)$ has odd density $1/2$ for all arithmetic progressions $Am + B$.
iii) If $\delta_F < 1/2$, then the coefficients of F are identically zero (mod 2) on some arithmetic progression.
iv) If the coefficients of F are not identically zero (mod 2) on any arithmetic progression, then they have odd density $1/2$ on every arithmetic progression; in particular, $\delta_F = 1/2$.
(Note: i), ii), and iii) together imply iv), and that iv) implies iii).)