$s \pmod{t}$-cores

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Outline

1. Background and context
2. $s \pmod{t}$-cores
3. Proofs for generating functions: the abacus
4. $C_{s(t)}(q)$
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Background and context

The *Ferrers diagram* of the partition $\lambda = (5, 3, 2, 2, 1, 1)$, with its *hooklengths* marked: the number of boxes directly right of and directly below the box, plus 1 for the box itself.

10	7	4	2	1
7	4	1		
5	2			
4	1			
2				
1				

This partition lacks hooks of length 3, 6, 8, 9, or anything larger than 10. A partition which lacks hooks of length t is t-*core*. Lacking hooks of length $t_1, t_2, \ldots$, a partition is a *simultaneous* $(t_1, t_2, \ldots)$-*core*.

Background and context

Let $f_t := \prod_{k=1}^{\infty}(1-q^{kt}) = (q^t;q^t)_\infty$.

Theorem (Olsson)

The generating function for the number $c_t(n)$ of t-core partitions of n is $\sum_{n=0}^{\infty} c_t(n) = \frac{f_t^t}{f_1}$.

Theorem (Aukerman, Kane, and Sze)

If $\gcd(s,t)=d$, the generating function for the number of (s,t)-cores is $C_{s,t}(q) = \sum_{n=0}^{\infty} c_{(s,t)}(n)q^n = \frac{f_d^d}{f_1} C_{s/d,t/d}(q^d)^d$.

Theorem (Anderson)

If s and t are coprime, the number of (s,t)-cores is $\frac{1}{s+t}\binom{s+t}{t}$.

Background and context

Relatively little is known about the polynomial factor $C_{s,t}(q)$ for general coprime s and t. We know its degree:

Theorem (Olsson and Stanton)

The largest (s, t)-core is of size $(s^2 - 1)(t^2 - 1)/24$.

For more indexes, the problem becomes rather wild; most results are known for $(t_1, \ldots, t_k)$ in arithmetic progression. Xiong (2016) gave the largest size of a $(s, s+1, \ldots, s+p)$-core, and Cho, Huh, and Sohn gave an enumeration of $(s, s+t, \ldots, s+pt)$-cores, to which we will return.

$s \pmod{t}$-cores

When the overall problem is wild, boundary cases can be of interest. Our project today is to look at $(s, s+t, \ldots, s+pt)$-cores in the large-p limit. If $s < t$, this is partitions in which no hook can be of length $s \pmod{t}$, so we might call these $s \pmod{t}$-cores, even if $s > t$. The t-cores are then $0 \pmod{t}$-cores.

$s \pmod{t}$-cores

Taking this boundary case allows us to establish more detailed results of combinatorial interest. We have the following general statement on their generating functions:

Theorem

If $gcd(s,t) = d$, the generating function for the number $c_{s(t)}(n)$ of $s \pmod{t}$-cores of n is

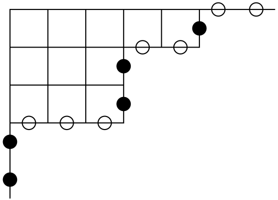
$$C_{s(t)}(q) = \sum_{n=0}^{\infty} c_{s(t)}(n) q^n = \frac{f_d^d}{f_1} C_{s/d(t/d)}(q^d)^d.$$

This looks very similar to the result for the (s,t)-cores, and is established similarly; only the polynomials are different.

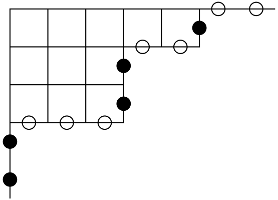
Proofs for generating functions: the abacus

In order to establish this and future theorems, we need the *abacus* of a partition.

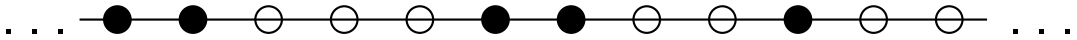
Mark the outer boundary of a partition with white “spacers” on the horizontal unit segments and black “beads” on the vertical segments. Allow for an indefinite extension of black beads prior to the diagram and white spacers afterward.



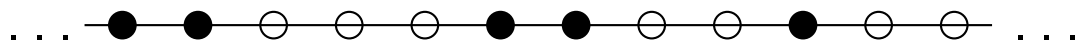
Proofs for generating functions: the abacus



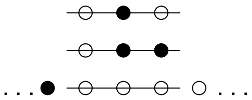
Now straighten out the profile. Since the partition starts with its first spacer and ends with its last bead, we have all information about the partition in its bead sequence.



Proofs for generating functions: the abacus

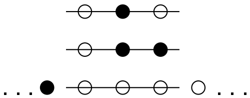


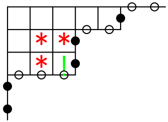
Finally, when we are interested in properties mod d, it is useful to fold the abacus back on itself, taking places d at a time to create the *d-runners*:



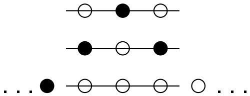
Manipulating the runners is the source of many proofs about cores.

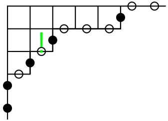
Proofs for generating functions: the abacus



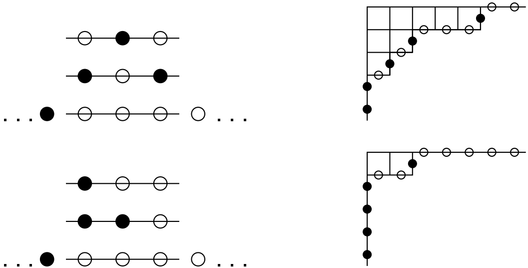


Moving a bead one place left on a runner changes two elements of the profile: a horizontal step followed by a vertical d places later to a vertical followed by a horizontal d places later. This removes a d-hook, exactly d squares in the diagram.





Proofs for generating functions: the abacus



If all beads are pushed as far left as possible, we have the diagram of the *d-core* of the partition. Every partition can thus be described in terms of its d-core, and the ordered d-tuple of *quotient* partitions represented by the diagrams described by each runner.

Proofs for generating functions: the abacus

Since the d distinct runners can be manipulated independently, and each is itself a partition with parts magnified by d, we have that the generating function of partitions equals the count of cores times d-tuples of d-magnified partitions, or,

$$\frac{1}{f_1} = \left(\sum_{n=0}^{\infty} c_d(n) q^n\right) \frac{1}{f_d^d} \Rightarrow \sum_{n=0}^{\infty} c_d(n) q^n = \frac{f_d^d}{f_1}.$$

Proofs for generating functions: the abacus

The same logic can be applied to proving that

Theorem

If gcd$(s,t) = d$, the generating function for the number of $s \pmod{t}$-cores is

$$C_{s(t)}(q) = \sum_{n=0}^{\infty} c_{s(t)}(n)q^n = \frac{f_d^d}{f_1} C_{s/d(t/d)}(q^d)^d.$$

Proof: Use the d-abacus. The cores are d-cores; there are d quotient partitions, magnified by d. In order to avoid hooks of length $s \pmod{t}$, it is necessary and sufficient that each quotient partition avoid hooks of length $s/d \pmod{(t/d)}$. $\square$

$s \pmod{t}$-cores: $C_{s(t)}(q)$

We can use the abacus to identify the polynomials $C_{s(t)}(q)$ for coprime s and t in small cases. For $s = 2$, t odd, we have:

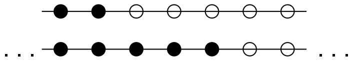
Theorem

$$C_{2(2m+1)}(q) = \sum_{n=0}^{\infty} c_{2(2m+1)}(n) q^n = \sum_{k=0}^{m+1} q^{\binom{k+1}{2}}.$$

Because there are so few degrees of freedom for 2-cores, this is the generating function for simultaneous $(2, 2m+3)$-cores. The bigger s is, the more conditions are necessary for equivalence.

$s \pmod{t}$-cores: $C_{s(t)}(q)$

For $C_{2(2m+1)}(q)$, look at the 2-abacus. We have a 0 runner and a 1 runner. A spacer followed by a bead is a hook of length 2, so once we have a spacer, we can have no more beads.



The partition starts with a spacer, so the 0 runner is the empty partition. The 1 runner can have beads up to some height, but cannot have a bead at position $2m+3$, or height $m+1$; the first spacer is at height k, from 0 to $m+1$. The available partitions are thus the "staircase" partitions $\{\varnothing, (1), (2,1), (3,2,1), \dots\}$ of sizes $\binom{n+1}{2}$, $n \geq 0$. The generating function is thus

$$C_{2(2m+1)}(q) = 1 + q + q^3 + q^6 + \cdots + q^{\binom{m+2}{2}}.$$

$s \pmod{t}$-cores: $C_{s(t)}(q)$

Here are the functions for $s = 3$:

Theorem

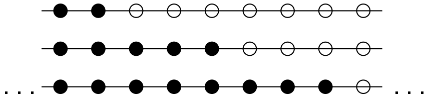
$$C_{3(3m+1)}(q) = -q^{(3m+5)(m+1)} + \sum_{j=0}^{m+1} q^{j^2+j} \sum_{\ell=-j}^{m+1} q^{\ell j+\ell^2+\ell}.$$

Theorem

$$C_{3(3m+2)}(q) = -q^{(3m+4)(m+2)} + \sum_{j=0}^{m+1} q^{j^2+j} \sum_{\ell=-j}^{m+2} q^{\ell j+\ell^2}.$$

$s \pmod{t}$-cores: $C_{s(t)}(q)$

The proof is similar. We look at the 3-abacus. The j and ℓ in the theorem are indexing the positions of the highest beads on the 1 and 2 runners. We avoid hooks of length 3 by filling with spacers as soon as we have one, and hooks of length $3m+4$ by avoiding a spacer followed by a bead on the next runner m places higher.



The proof is now simply a matter of counting the contributed parts when the heights on the runners are in given position. The subtracted term simplifies the sums (and, if not subtracted, we would have the simultaneous $(3, 3m+4)$-cores).

$s \pmod{t}$-cores: $C_{s(t)}(q)$

And so between these two ingredients we can write down lots of specific generating functions:

$$C_{2(5)}(q) = 1 + q + q^3 + q^6$$

$$C_{4(10)}(q) = \frac{f_2^2}{f_1}(C_{2(5)}(q^2))^2 = \left(\sum_{n=0}^{\infty} q^{\binom{n+1}{2}}\right)(1 + q^2 + q^6 + q^{12})^2$$

$$C_{14(35)}(q) = \frac{f_7^7}{f_1}(1 + q^7 + q^{42} + q^{84})^7$$

$$C_{3(4)}(q) = 1 + q + 2q^2 + 2q^4 + q^5 + 2q^6 + q^8 + 2q^9 + 2q^{10}$$

$$C_{6(9)}(q) = \left(\frac{f_3^3}{f_1}\right)\left(1 + q^3 + q^9\right)^3$$

$$C_{6(8)}(q) = \left(\frac{f_2^2}{f_1}\right)\left(C_{3(4)}(q^2)\right)^2.$$

Congruences

Congruences for a sequence $\{g(n)\}$, of the form $g(An + B) \equiv 0 \pmod{d}$, are a popular form of result in partition theory. They tend to arise as symmetries of eta-quotients, i.e. from the properties of weakly holomorphic modular forms applied to functions of the form $\prod_{i=1}^{\infty} f_i^{t_i}$, $t_i \in \mathbb{Z}$. The t-cores, for instance, have many.

Since the functions we're considering are d-cores *times polynomials*, the entries in their sequence of coefficients are linear combinations of eta-quotient coefficients, so it's perhaps interesting that we still find such congruences.

Congruences

If $gcd(s, t) = 2$, we have a special case, since $\frac{f_2^2}{f_1} = \sum_{n=0}^{\infty} q^{\binom{n+1}{2}}$; multiplied times a polynomial, one gets a finite sequence of nonzero values at increasing intervals, so the coefficients are mostly zero.

For $gcd(s, t) = 3$, the 3-core factor is much denser, so the following congruences are considerably more nontrivial:

Theorem

We have that

$$c_{6(9)}(16n + 9) \equiv 0 \pmod{2},$$

and for $k \in \{38, 88, 118, 168, 198, 248, 278, 328\}$, *that*

$$c_{6(9)}(400n + k) \equiv 0 \pmod{3}.$$

Congruences: proof of $c_{6(9)}(16n+9) \equiv 0 \pmod 2$

The following is a known identity:

$$\frac{f_3^3}{f_1} \equiv_2 \sum_{n\in\mathbb{Z}} q^{n(3n-2)}.$$

Now observe quadratic residues. We have that $n(3n-2)$ may take residues 0, 1, 5, or 8 modulo 16, and the polynomial multiplier is

$$\left(1+q^3+q^9\right)^3 \equiv_2 1+q^3+q^6+q^{15}+q^{18}+q^{21}+q^{27}.$$

Among the possible residues modulo 16 for

$$\{0,1,5,8\}+\{0,2,3,5,6,11,15\},$$

the sum 9 modulo 16 does not appear. $\square$

Congruences: proof of $c_{6(9)}(400n + k) \equiv 0 \pmod 3$

We begin with a known dissection for $\frac{f_9}{f_1}$:

$$\frac{f_3^3}{f_1} \equiv_3 \frac{f_9}{f_1} \equiv_3 f_2^4 + q\frac{f_{36}}{f_4} \equiv_3 f_2^4 + qf_8^4 + q^5\frac{f_{144}}{f_{16}}.$$

Likewise, we observe that

$$(1 + q^3 + q^9)^3 \equiv_3 1 + q^9 + q^{27}.$$

Since f_{144}/f_{16} is a function of q^{16}, nonzero terms in

$$q^5\frac{f_{144}}{f_{16}}(1 + q^9 + q^{27})$$

will appear with residue 5, 14, or 0 modulo 16; the residues k listed in the theorem as avoided are all 6 or 8 modulo 16. Likewise for $qf_8^4(1 + q^9 + q^{27})$.

Congruences: proof of $c_{6(9)}(400n+k) \equiv 0 \pmod 3$

The k are all even so we end up only needing to worry about f_2^4.

We now use Euler's pentagonal number theorem to write

$$f_2^4 \equiv_3 f_6 f_2 \equiv_3 \left(\sum_{m \in \mathbb{Z}} (-1)^m q^{(3m)(3m-1)} \right) \left(\sum_{n \in \mathbb{Z}} (-1)^n q^{(n)(3n-1)} \right).$$

We observe the residues modulo 400 of $9m^2 - 3m$ and $3n^2 - n$, see that the claimed residues are missed by their sums, and the theorem is proved. □

Fayers' conjecture

So far we have been working with $s \pmod{t}$-cores enumerated with respect to their size, and focusing on cases where s is not coprime to t. The polynomial factors arise from the coprime cases, and in these cases the number of $s \pmod{t}$-cores is finite.

The following theorem of Cho, Huh, and Sohn enumerates simultaneous $(s, s+t, s+2t, \ldots, s+pt)$-core partitions:

Theorem (Cho, Huh, and Sohn)

If $d = 1$, then the number of $(s, s+t, s+2t, \ldots, s+pt)$-core partitions is

$$\frac{1}{s+t}\binom{s+t}{t}+\sum_{k=1}^{\lfloor s/2 \rfloor} \sum_{\ell=0}^{r} \frac{1}{k+t}\binom{k+t}{k-\ell}\binom{k-1}{\ell}\binom{s+t-\ell(p-2)-1}{2k+t-1}.$$

Fayers' conjecture

In the large-p limit, simultaneous $(s, s+t, s+2t, \ldots, s+pt)$-core partitions become $s \pmod{t}$-cores. In the large-p limit, their formula becomes

$$
\begin{aligned}
g_t(s) &= \frac{1}{s+t}\binom{s+t}{t} + \sum_{k=1}^{\lfloor s/2 \rfloor} \frac{1}{k+t}\binom{k+t}{k}\binom{s+t-1}{2k+t-1} \\
&= \sum_{k=0}^{\lfloor s/2 \rfloor} \frac{1}{t}\binom{k+t-1}{t-1}\binom{s+t-1}{2k+t-1} \\
&= \sum_{n=0}^{\infty} c_{s(t)}(n).
\end{aligned}
$$

Fayers' conjecture

A conjecture of Matt Fayers suggests that this formula grows as an exponential in s times a polynomial dependent on t:

Conjecture (Fayers)

Suppose $t \geq 1$. Then there is a monic polynomial $f_t(s)$ of degree $t-1$ with non-negative integer coefficients such that for any $s \geq 1$ coprime to t, the number of $(s, s+t, s+2t, \ldots)$-cores is

$$\frac{2^{s-t} f_t(s)}{t!}.$$

The constant term of $f_t(s)$ is $(2^t - 1)(t-1)!$.

Fayers' conjecture

We are able to prove this conjecture and identify the polynomials:

Theorem

Conjecture 1 holds. The polynomials required are given by

$$f_t(s) = \sum_{m=1}^{t} L(t,m)(s+1)^{\overline{m-1}} \tag{1}$$

where the $L(t,m)$ are the unsigned Lah numbers defined for $t \geq m \geq 1$ by

$$L(t,m) = \frac{t!}{m!}\binom{t-1}{m-1}$$

with $x^{\overline{0}} = 1$ for $x \geq 0$ and, for $m \geq 1$, $x^{\overline{m}} := x(x+1)(x+2)\dots(x+m-1)$. (The $x^{\overline{m}}$ are the rising factorials.)

Fayers' conjecture

Define

$$g_t^*(s) = \frac{2^{s-t}}{t!} \sum_{m=1}^{t} L(t,m)(s+1)^{\overline{m-1}}.$$

We would like this to be equal to Cho, Huh, and Sohn's $g_t(s)$.

One may by hand or by use of Maple's *sumtools* and *sumrecursion* commands verify that $g_t(s)$ satisfies the second-order recurrence

$$2tg_t(s) = (s+3(t-1))g_{t-1}(s) - (t-2)g_{t-2}(s)$$

with initial conditions

$$g_1(s) = \sum_{k=0}^{\lfloor s/2 \rfloor} \binom{s}{2k} \quad , \quad g_2(s) = \sum_{k=0}^{\lfloor s/2 \rfloor} \frac{k+1}{2} \binom{s+1}{2k+1}.$$

Fayers' conjecture

The full $g_t^*(s)$ is a bit much for Maple to chew, but the polynomials $f_t(s) = f_t(s) = \sum_{m=1}^{t} L(t, m)(s+1)^{\overline{m-1}}$ can also be analyzed. They satisfy the second-order recurrence

$$f_t(s) = (s + 3(t-1))f_{t-1}(s) - 2(t-1)(t-2)f_{t-2}(s)$$

with initial conditions

$$f_1(s) = L(1,1)(s+1)^{\overline{0}} = 1 \cdot 1 = 1$$

and

$$f_2(s) = L(2,1)(s+1)^{\overline{0}} + L(2,2)(s+1)^{\overline{1}} = 2 \cdot 1 + 1 \cdot (s+1) = s+3.$$

Fayers' conjecture

Rewrite this recurrence with $g_t^*(s)$ to get

$$g_t^*(s) = (s + 3(t-1))\frac{(t-1)!}{2^{s-(t-1)}} \cdot \frac{2^{s-t}}{t!} g_{t-1}^*(s) - 2(t-1)(t-2)\frac{(t-2)!}{2^{s-(t-2)}} \cdot \frac{2^{s-t}}{t!} g_{t-2}^*(s).$$

Cross-multiplying and cancelling factors from the factorials, we find that this is equivalent to

$$2tg_t^*(s) = (s + 3(t-1))g_{t-1}^*(s) - (t-2)g_{t-2}^*(s),$$

as desired.

Fayers' conjecture

We still need to show that initial conditions match.
We have that $g_1^*(s) = \frac{2^{s-1}}{1!}(1) = 2^{s-1}$. We calculate

$$g_1(s) = \sum_{k=0}^{\lfloor s/2 \rfloor} \binom{s}{2k} = \sum_{k=0}^{\lfloor s/2 \rfloor} \binom{s-1}{2k-1} + \binom{s-1}{2k}$$
$$= \sum_{j=0}^{s-1} \binom{s-1}{j} = 2^{s-1}.$$

Fayers' conjecture

For $t = 2$, we have that

$$g_2^*(s) = 2^{s-3}(s+3) \quad , \quad g_2(s) = \sum_{k=0}^{\lfloor s/2 \rfloor} \frac{k+1}{2} \binom{s+1}{2k+1}.$$

Fayers' conjecture

Recall the Binomial Theorem which states that, for all $n \geq 0$,

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Hence

$$\frac{(1+x)^{s+1} - (1-x)^{s+1}}{2} = \sum_{k \geq 0} \binom{s+1}{2k+1} x^{2k+1}.$$

We differentiate this expression with respect to x to obtain

$$\frac{1}{2}\left((s+1)(1+x)^s + (s+1)(1-x)^s\right) = \sum_{k \geq 0} (2k+1) \binom{s+1}{2k+1} x^{2k}.$$

Fayers' conjecture

Now sum the previous two lines to obtain

$$\frac{(1+x)^{s+1}-(1-x)^{s+1}}{2}+\frac{(s+1)(1+x)^s+(s+1)(1-x)^s}{2}$$
$$=\sum_{k\geq 0}\left[x\binom{s+1}{2k+1}+(2k+1)\binom{s+1}{2k+1}\right]x^{2k}.$$

Set $x=1$ in this expression to obtain

$$\frac{2^{s+1}+(s+1)2^s}{2}=\sum_{k\geq 0}(2k+2)\binom{s+1}{2k+1}.$$

Divide through by 4 to obtain

$$g_2(s)=\sum_{k\geq 0}\frac{k+1}{2}\binom{s+1}{2k+1}=2^{s-3}(s+3),$$

as desired.

Fayers' conjecture

The polynomial $f_t(s)$ is clearly monic and degree $t-1$. His last claim is that the constant term is $(2^t-1)(t-1)!$. We have

$$\begin{aligned} CT\left(f_t(s)\right) &= \sum_{m=1}^{t} L(t,m)(m-1)! \\ &= \sum_{m=1}^{t} \frac{t!}{m!}\binom{t-1}{m-1}(m-1)! \\ &= (t-1)! \sum_{m=1}^{t} \frac{t!}{m!(t-m)!} \\ &= (t-1)! \sum_{m=1}^{t} \binom{t}{m} \\ &= (t-1)!(2^t-1). \end{aligned}$$

Further questions

1. We can certainly play the residue game with other $C_{s(t)}$ and the associated cores. Can we get infinite families of congruences? Any Ramanujan-like congruence for the partition function, $p(An+B) \equiv 0 \pmod{d}$ with $d|A$, will be inherited by any $C_{JA,KA}(q)$, and likewise from A-cores, so "new" congruences seem the more interesting.
2. Consider $C_{3(3m+1)}(q)$ and $C_{3(3m+2)}(q)$. As $m \to \infty$, both of these functions approach the 3-cores, f_3^3/f_1 (in different ways). Combinatorially, the sets of 3 $\pmod{(3m+i)}$-cores are distinct, some contain each other, and they eventually exhaust the set of 3-cores in a hierarchy. Is there any algebraically or structurally interesting feature of this division?
3. Fayers had one more clause in his conjecture we were unable to prove: that $f_t(s)$ is divisible by $s+t+(-1)^t$. At the moment, that is still open.