Rogers Ramanujan Gordon Identities when $k = 3$

Yalçın Can Kılıç

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Outline

Definitions

Let n be a natural number. Then, an **integer partition** of n is a sequence $(\lambda_1, \lambda_2, \cdots, \lambda_k)$ such that λ_i is a positive integer for all i, $\lambda_1 + \lambda_2 + \cdots \lambda_k = n$. Moreover, $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_k \geq 1$. We call each λ_i a **part**. Let $p(n)$ be the number partitions of n. If λ is a partition of n, we denote it by $|\lambda| = n$ and n is called the **size** of λ.

Example

Let $n = 4$. Then, all partitions of 4 can be listed as follows: 4, $3+1$, $2+2$, $2+1+1$, $1+1+1+1$. Thus, $p(4) = 5$.

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Usually, we want to consider a certain subset of integer partitions specified by some conditions. Let's look at some examples:

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Let $n = 5$. All partitions of 5 where each part is odd can be listed as: 5 , $3+1+1$, $1+1+1+1+1$.

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Similarly, all partitions of 5 where parts are distinct can be listed as: 5 , $4+1$ and $3+2$.

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Any identity of the form

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First partition identity is due to Euler:

Theorem (Euler's Identity)

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Generating Functions

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A **combinatorial class** $\mathcal{A}$ is a countable set on a which a *size function* , $|.| : \mathcal{A} \to \mathbb{N}$, is defined, satisfying the following conditions:

1. The size of an element is a nonnegative integer
2. The number of elements of any given size is finite.

Definition

Let $\mathcal{A}$ be a combinatorial class, the **generating function** for $\mathcal{A}$ is

$$A(q) := \sum_{\alpha \in \mathcal{A}} q^{|\alpha|}$$

For this talk, we see $A(q)$ as a formal object, i.e there is no convergence issues.

Central Idea: To prove a partition identity, show that their generating functions for both sides are the same.

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$q-$Pochammer Symbol

Before proving Euler's Identity, let's define $q-$Pochammer symbol:
Let $n \geq 0$. Then,

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

Also, $(a;q)_0 := 1$.

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Generating Functions for Partitions

Let's look at some generating function examples. Let $\mathcal{P}$ be the combinatorial class which contains all integer partitions.

Example

1. The generating function for partitions in which each part is at most m:

$$\sum_{\lambda\in\mathcal{P},\text{ parts in }\{1,2,\cdots,m\}} q^{|\lambda|} = 1+q+2q^2+\cdots = \frac{1}{1-q}\frac{1}{1-q^2}\cdots\frac{1}{1-q^m}$$

2. The generating function for partitions without any restrictions is:

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Proof of Euler's Identity

Let's show that generating function of odd partitions and distinct partitions are the same:

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\sum_{\text{all odd partitions } \lambda} q^{|\lambda|} &= \prod_{n\geq 1}^{\infty} \frac{1}{1-q^{2n-1}} \\
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General View on Euler's Identity

Let's recall Euler's Identity:

Theorem (Euler's Identity)

Let n be a nonnegative integer. Then,

$$p(n|\text{all parts are odd}) = p(n|\text{all parts are distinct})$$

On the left hand side, we have a modulus condition, on the right hand side we have a difference condition.
This is a very general family of partitions as we will see in the following slide.

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Difference Conditions-Modulus Conditions

Let's look at some famous theorems of the form:

$$p(n|\text{difference conditions}) = p(n|\text{modulus conditions})$$

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Theorem (Rogers-Ramanujan 1)

Let n be any natural number. Let $A(n)$ be the number of partitions of n into parts such that the consecutive differences between the parts are at least 2. Let $B(n)$ be the number of partitions of n into parts where each part is $\equiv \pm 1 \mod 5$. Then, $A(n) = B(n)$ for all n.

Theorem (Rogers-Ramanujan 2)

Let n be any natural number. Let $A(n)$ be the number of partitions of n into parts such that the consecutive differences between the parts are at least 2 and 1 does not appear as a part. Let $B(n)$ be the number of partitions of n into parts where each part is $\equiv \pm 2 \mod 5$. Then, $A(n) = B(n)$ for all n.

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Gordon's Generalization

Gordon generalizes Rogers-Ramanujan identities as follows:

Theorem (Rogers-Ramanujan-Gordon,1961)

Let a and k be natural numbers such that $1 \leq a \leq k$. Then, the number of partitions of n into parts not equivalent to $0, \pm a \mod 2k+1$ is equal to the number of partitions of $n = \lambda_1 + \lambda_2 + \cdots + \lambda_m$ where $\lambda_i \geq \lambda_{i+k-1} + 2$ and the number of 1's are at most $a-1$.

Remark

$(k, a) = (2, 1)$ corresponds to Rogers-Ramanujan 1. Similarly, $(k, a) = (2, 2)$ corresponds to Rogers-Ramanujan 2.

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Theorem (Rogers-Ramanujan-Gordon,1961)

Let a and k be natural numbers such that $1 \leq a \leq k$*. Then, the number of partitions of n into parts not equivalent to* $0, \pm a \mod 2k+1$ *is equal to the number of partitions of* $n = \lambda_1 + \lambda_2 + \cdots + \lambda_m$ *where* $\lambda_i \geq \lambda_{i+k-1} + 2$ *and the number of* 1*'s are at most* $a - 1$*.*

Remark

$(k, a) = (2, 1)$ corresponds to Rogers-Ramanujan 1. Similarly, $(k, a) = (2, 2)$ corresponds to Rogers-Ramanujan 2.

An Example

Example

Let $(k, a) = (3, 2)$, i.e the difference should be at least 2 in distance 2 and we can use at most one 1 as a part, and $n = 7$. Then, on the modulus side we have $6 + 1$, $4 + 3$, $4 + 1 + 1 + 1$, $3 + 3 + 1$, $3 + 1 + 1 + 1 + 1$, $1 + 1 + 1 + 1 + 1 + 1 + 1$. Similarly, on the difference side we have 7 , $6 + 1$, $5 + 2$, $4 + 3$, $4 + 2 + 1$, $3 + 3 + 1$.

Analytic Version of Rogers-Ramanujan-Gordon

Andrews found the corresponding version of Rogers-Ramanujan-Gordon identities:

Theorem (Andrews,1974)

Let $1 \leq a \leq k$ be integers. Then,

$$\sum_{n_1,n_2,\cdots,n_{k-1}\geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_a+N_{a+1}+\cdots+N_{k-1}}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{k-1}}} = \prod_{\substack{n=1 \\ n\not\equiv 0,\pm a}}^{\infty} \frac{1}{1-q^n}$$

where $N_i := n_i + n_{i+1} + \cdots + n_{k-1}$.

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where $N_i := n_i + n_{i+1} + \cdots + n_{k-1}$.

Notation

Let a, k, m, n be nonnegative integers such that $1 \leq a \leq k$. Then, we define $rrg_{k,a}(m, n)$ as follows: Number of partitions of n into m parts, $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_m$ such at most $a - 1$ parts are 1 and the difference at distance k is at least 2, i.e $\lambda_i \geq \lambda_{i+k-1} + 2$.

Andrews-Gordon Series when $k = 3$

If forget about the modulus side of the Andrew-Gordon series, take number of parts into account and specialize for $(k, a) = (3, 3)$, we get:

$$\sum_{m,n\geq 0} rrg_{3,3}(m,n)q^n x^m = \sum_{m,n\geq 0} \frac{q^{4\binom{m+1}{2}+2mn+2\binom{n+1}{2}-2m-n}x^{2m+n}}{(q;q)_m(q;q)_n}$$

A New Series for RRG when k=3

Our series for RRG $k = 3$ is as follows:

Theorem (YCK, 2023)

1. $$\sum_{n,m\geq 0} rrg_{3,1}(m,n)q^n x^m = \sum_{m,n\geq 0} \frac{q^{4\binom{m+1}{2}+2mn+\binom{n+1}{2}+n} x^{2m+n}}{(q^2;q^2)_m (q;q)_n}$$

2. $$\sum_{n,m\geq 0} rrg_{3,2}(m,n)q^n x^m = \sum_{m,n\geq 0} \frac{q^{4\binom{m+1}{2}+2mn+\binom{n+1}{2}} x^{2m+n}}{(q^2;q^2)_m (q;q)_n}$$

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Intuitive Ideas

Suppose we are given an evidently positive generating function of the form

$$\sum_{m,n\geq 0} \frac{q^{QUADRATIC+LINEAR} x^{LINEAR}}{(q^B;q^C)_m (q^D;q^E)_n}$$

Then, the numerator corresponds to the *base partition* and the denominator corresponds to the moves on each part. Thus, given a partition which satisfies the properties, we will find the smallest weight partition that satisfies the conditions and using moves construct the given partition.

Given a partition which satisfies the conditions of $rrg_{3,3}$, our aim is to construct it starting from the *base partition* and using *moves* on the parts.

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Examples

Let's look at an example:

$\lambda = 14 + 14 + 11 + 7 + 7 + 2 + 1.$

Our aim is to construct a partition triple (β, μ, ν) where β is the base partition, μ is the partition which contains backward moves applied on the pairs of λ and ν is the partition which contains backward moves applied on the singletons.

Side Note: We allow 0 as a part in μ and ν.

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Step 1

Firstly, we divide the parts into two: The ones which repeats(pairs) and the ones which do not repeat(singletons).
$\lambda = [14, 14] + (11) + [7, 7] + (2) + (1)$. Now, we are looking for a partition which contains $m = 2$ pairs and $n = 3$ singletons.

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Now, we will answer the following question: What is the smallest weight partition that contains 2 pairs and 3 singletons?

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We want to reach $\beta = (7) + (6) + (5) + [3,3] + [1,1]$ from $\lambda = [14,14] + (11) + [7,7] + (2) + (1)$.

Firstly, we will obtain $[1,1]$ using backward moves:

$$\begin{aligned}[14,14] + (11) + [7,7] + (2) + (1) &\to [14,14] + (11) + \mathbf{[6,6]} + (2) + (1)\\ &\to [14,14] + (11) + \mathbf{[5,5]} + (2) + (1)\\ &\to [14,14] + (11) + \mathbf{[4,4]} + (2) + (1)\end{aligned}$$

Now, we want to pull back $[4,4]$ to get $[3,3]$ however the resulting partition does not satisfy the conditions $rrg_{3,3}$. Thus, we need to do some arrangements on the parts.

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Step3(Continued)

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We did obtained $[3,3]$. We are finished with the pairs. Now, we need to obtain singletons.

Step 3 Continued

Let's recall $\beta = (7) + (6) + (5) + [3,3] + [1,1]$. Now, we will obtain the singletons, (5) , (6) and (7) in this order. However, (5) and (6) are already present! We just need to obtain (7):

$$\begin{aligned}(13) + (6) + (5) + [3,3] + [1,1] &\rightarrow \mathbf{(12)} + (6) + (5) + [3,3] + [1,1] \\ &\rightarrow \mathbf{(11)} + (6) + (5) + [3,3] + [1,1] \\ &\rightarrow \mathbf{(10)} + (6) + (5) + [3,3] + [1,1] \\ &\rightarrow \mathbf{(9)} + (6) + (5) + [3,3] + [1,1] \\ &\rightarrow \mathbf{(8)} + (6) + (5) + [3,3] + [1,1] \\ &\rightarrow \mathbf{(7)} + (6) + (5) + [3,3] + [1,1]\end{aligned}$$

We are done!

Step 3 Continued

Let's recall $\beta = (7) + (6) + (5) + [3,3] + [1,1]$. Now, we will obtain the singletons, (5) , (6) and (7) in this order. However, (5) and (6) are already present! We just need to obtain (7):

$$\begin{aligned}(13) + (6) + (5) + [3,3] + [1,1] &\to \mathbf{(12)} + (6) + (5) + [3,3] + [1,1]\\ &\to \mathbf{(11)} + (6) + (5) + [3,3] + [1,1]\\ &\to \mathbf{(10)} + (6) + (5) + [3,3] + [1,1]\\ &\to \mathbf{(9)} + (6) + (5) + [3,3] + [1,1]\\ &\to \mathbf{(8)} + (6) + (5) + [3,3] + [1,1]\\ &\to \mathbf{(7)} + (6) + (5) + [3,3] + [1,1]\end{aligned}$$

We are done!

Step 3 Continued

Let's recall $\beta = (7) + (6) + (5) + [3,3] + [1,1]$. Now, we will obtain the singletons, (5) , (6) and (7) in this order. However, (5) and (6) are already present! We just need to obtain (7):

$$
\begin{aligned}
(13) + (6) + (5) + [3,3] + [1,1] &\rightarrow \mathbf{(12)} + (6) + (5) + [3,3] + [1,1] \\
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&\rightarrow \mathbf{(9)} + (6) + (5) + [3,3] + [1,1] \\
&\rightarrow \mathbf{(8)} + (6) + (5) + [3,3] + [1,1] \\
&\rightarrow \mathbf{(7)} + (6) + (5) + [3,3] + [1,1]
\end{aligned}
$$

We are done!

Step 3 Continued

Let's recall $\beta = (7) + (6) + (5) + [3,3] + [1,1]$. Now, we will obtain the singletons, (5) , (6) and (7) in this order. However, (5) and (6) are already present! We just need to obtain (7):

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(13) + (6) + (5) + [3,3] + [1,1] &\to \mathbf{(12)} + (6) + (5) + [3,3] + [1,1] \\
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Let's recall $\beta = (7) + (6) + (5) + [3,3] + [1,1]$. Now, we will obtain the singletons, (5) , (6) and (7) in this order. However, (5) and (6) are already present! We just need to obtain (7):

$$\begin{aligned}(13) + (6) + (5) + [3,3] + [1,1] &\to \mathbf{(12)} + (6) + (5) + [3,3] + [1,1] \\ &\to \mathbf{(11)} + (6) + (5) + [3,3] + [1,1] \\ &\to \mathbf{(10)} + (6) + (5) + [3,3] + [1,1] \\ &\to \mathbf{(9)} + (6) + (5) + [3,3] + [1,1] \\ &\to \mathbf{(8)} + (6) + (5) + [3,3] + [1,1] \\ &\to \mathbf{(7)} + (6) + (5) + [3,3] + [1,1]\end{aligned}$$

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Let's recall $\beta = (7) + (6) + (5) + [3,3] + [1,1]$. Now, we will obtain the singletons, (5) , (6) and (7) in this order. However, (5) and (6) are already present! We just need to obtain (7):

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We are done!

Step 4

Let's store the number of moves we applied on each singleton and each pairs:

We applied 4 backward moves on the pair $[7, 7]$ and 8 backward moves on the pair $[13, 13]$. Thus, $\mu = (8 \times 2, 4 \times 2)$.

Similarly, we applied 0 backward moves on the singleton (1), 0 backward moves on the singleton (2) and 6 backward moves on the singleton (11). Hence, $\nu = (6, 0, 0)$.

As a result, we obtained the following correspondence

$$[14, 14] + (11) + [7, 7] + (2) + (1) \to$$
$$(\beta = (7) + (6) + (5) + [3, 3] + [1, 1],\ \mu = (8 \times 2, 4 \times 2),\ \nu = (6, 0, 0))$$

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Examples

Now, we want get the other direction of the correspondence, i.e given (β, μ, ν) obtain λ:

Write them below using the above example. Let $\beta = (7) + (6) + (5) + [3,3] + [1,1]$, $\mu = (8 \times 2, 4 \times 2)$ and $\nu = (6,0,0)$

Examples

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Write them below using the above example. Let
$\beta = (7) + (6) + (5) + [3, 3] + [1, 1]$, $\mu = (8 \times 2, 4 \times 2)$ and $\nu = (6, 0, 0)$

Step 1

This time we will start with the singletons. First, we will apply the forward moves on the singletons using $\nu = (6, 0, 0)$. In other words, we will apply 6 forward moves on (7) and no other forward moves on (6) and (5).

$$\begin{aligned}(7) + (6) + (5) + [3, 3] + [1, 1] &\to \mathbf{(8)} + (6) + (5) + [3, 3] + [1, 1] \\ &\to \mathbf{(9)} + (6) + (5) + [3, 3] + [1, 1] \\ &\to \mathbf{(10)} + (6) + (5) + [3, 3] + [1, 1] \\ &\to \mathbf{(11)} + (6) + (5) + [3, 3] + [1, 1] \\ &\to \mathbf{(12)} + (6) + (5) + [3, 3] + [1, 1] \\ &\to \mathbf{(13)} + (6) + (5) + [3, 3] + [1, 1]\end{aligned}$$

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\begin{aligned}
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$$\begin{aligned}
(7) + (6) + (5) + [3,3] + [1,1] &\to \mathbf{(8)} + (6) + (5) + [3,3] + [1,1] \\
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&\to \mathbf{(13)} + (6) + (5) + [3,3] + [1,1]
\end{aligned}
$$

Step 2

Now, we will forward moves on the pairs using $\mu = (8 \times 2, 4 \times 2)$, i.e we will push the pair $[3,3]$ 8 times and the pair $[1,1]$ 4 times:

$$
\begin{aligned}
(13)+(6)+(5)+[3,3]+[1,1] &\to (13)+\mathbf{[6,6]}+(4)+(3)+[1,1] \\
&\to (13)+\mathbf{[7,7]}+(4)+(3)+[1,1] \\
&\to (13)+\mathbf{[8,8]}+(4)+(3)+[1,1] \\
&\to (13)+\mathbf{[9,9]}+(4)+(3)+[1,1] \\
&\to (13)+\mathbf{[10,10]}+(4)+(3)+[1,1] \\
&\to (13)+\mathbf{[11,11]}+(4)+(3)+[1,1] \\
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We are finished with $[3,3]$.

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&\to (13)+\mathbf{[10,10]}+(4)+(3)+[1,1] \\
&\to (13)+\mathbf{[11,11]}+(4)+(3)+[1,1] \\
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\begin{aligned}
(13) + (6) + (5) + [3,3] + [1,1] &\to (13) + \textbf{[6,6]} + (4) + (3) + [1,1] \\
&\to (13) + \textbf{[7,7]} + (4) + (3) + [1,1] \\
&\to (13) + \textbf{[8,8]} + (4) + (3) + [1,1] \\
&\to (13) + \textbf{[9,9]} + (4) + (3) + [1,1] \\
&\to (13) + \textbf{[10,10]} + (4) + (3) + [1,1] \\
&\to (13) + \textbf{[11,11]} + (4) + (3) + [1,1] \\
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$$\begin{aligned}(13)+(6)+(5)+[3,3]+[1,1] &\to (13)+\mathbf{[6,6]}+(4)+(3)+[1,1]\\ &\to (13)+\mathbf{[7,7]}+(4)+(3)+[1,1]\\ &\to (13)+\mathbf{[8,8]}+(4)+(3)+[1,1]\\ &\to (13)+\mathbf{[9,9]}+(4)+(3)+[1,1]\\ &\to (13)+\mathbf{[10,10]}+(4)+(3)+[1,1]\\ &\to (13)+\mathbf{[11,11]}+(4)+(3)+[1,1]\\ &\to \mathbf{[13,13]}+(11)+(4)+(3)+[1,1]\\ &\to \mathbf{[14,14]}+(11)+(4)+(3)+[1,1]\end{aligned}$$

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\end{aligned}$$

We are finished with $[3,3]$.

Step 2

Now, we will forward moves on the pairs using $\mu = (8 \times 2, 4 \times 2)$, i.e we will push the pair $[3,3]$ 8 times and the pair $[1,1]$ 4 times:

$$
\begin{aligned}
(13)+(6)+(5)+[3,3]+[1,1] &\to (13)+\mathbf{[6,6]}+(4)+(3)+[1,1]\\
&\to (13)+\mathbf{[7,7]}+(4)+(3)+[1,1]\\
&\to (13)+\mathbf{[8,8]}+(4)+(3)+[1,1]\\
&\to (13)+\mathbf{[9,9]}+(4)+(3)+[1,1]\\
&\to (13)+\mathbf{[10,10]}+(4)+(3)+[1,1]\\
&\to (13)+\mathbf{[11,11]}+(4)+(3)+[1,1]\\
&\to \mathbf{[13,13]}+(11)+(4)+(3)+[1,1]\\
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$$
\begin{aligned}
(13) + (6) + (5) + [3,3] + [1,1] &\to (13) + \mathbf{[6{,}6]} + (4) + (3) + [1,1] \\
&\to (13) + \mathbf{[7{,}7]} + (4) + (3) + [1,1] \\
&\to (13) + \mathbf{[8{,}8]} + (4) + (3) + [1,1] \\
&\to (13) + \mathbf{[9{,}9]} + (4) + (3) + [1,1] \\
&\to (13) + \mathbf{[10{,}10]} + (4) + (3) + [1,1] \\
&\to (13) + \mathbf{[11{,}11]} + (4) + (3) + [1,1] \\
&\to \mathbf{[13{,}13]} + (11) + (4) + (3) + [1,1] \\
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$$
\begin{aligned}
(13) + (6) + (5) + [3,3] + [1,1] &\rightarrow (13) + \mathbf{[6,6]} + (4) + (3) + [1,1] \\
&\rightarrow (13) + \mathbf{[7,7]} + (4) + (3) + [1,1] \\
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\end{aligned}$$

We are finished with $[3,3]$.

Step 2(Continued)

Now, we will push the pair $[1,1]$:

$$\begin{aligned}[14,14]+(11)+(4)+(3)+[1,1] &\rightarrow [14,14]+(11)+\mathbf{[4,4]}+(2)+(1)\\ &\rightarrow [14,14]+(11)+\mathbf{[5,5]}+(2)+(1)\\ &\rightarrow [14,14]+(11)+\mathbf{[6,6]}+(2)+(1)\\ &\rightarrow [14,14]+(11)+\mathbf{[7,7]}+(2)+(1)\end{aligned}$$

We are finished with the pairs as well.

Step 2(Continued)

Now, we will push the pair $[1, 1]$:

$$
\begin{aligned}
[14,14] + (11) + (4) + (3) + [1,1] &\to [14,14] + (11) + \mathbf{[4,4]} + (2) + (1) \\
&\to [14,14] + (11) + \mathbf{[5,5]} + (2) + (1) \\
&\to [14,14] + (11) + \mathbf{[6,6]} + (2) + (1) \\
&\to [14,14] + (11) + \mathbf{[7,7]} + (2) + (1)
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$$
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&\rightarrow [14, 14] + (11) + \mathbf{[5,5]} + (2) + (1) \\
&\rightarrow [14, 14] + (11) + \mathbf{[6,6]} + (2) + (1) \\
&\rightarrow [14, 14] + (11) + \mathbf{[7,7]} + (2) + (1)
\end{aligned}
$$

We are finished with the pairs as well.

Step 3

As a result, we showed the following correspondence:

$$(\beta = (7) + (6) + (5) + [3,3] + [1,1],\ \mu = (8 \times 2, 4 \times 2),\ \nu = (6,0,0)) \to [14,14] + (11) + [7,7] + (2) + (1)$$

Main Theorem

Hence, we showed that

Theorem

$$\sum_{n,m\geq 0} rrg_{3,3}(m,n)q^n x^m = \sum_{m,n\geq 0} \frac{q^{4\binom{m+1}{2}+2mn+\binom{n+1}{2}-2m} x^{2m+n}}{(q^2;q^2)_m (q;q)_n}$$

Sketch of the Proof

The proof contains 3 steps:

1. A lemma about the form of the base partition
2. A general form of the moves
    1. Forward moves
    2. Backward moves
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A lemma about Base Partitions

We will state the lemma about the base partition now:

Lemma (Form of the Base Partition)

The base partition for $rrg_{3,3}$ with m pairs and n singletons is

$$(2m+n)+\cdots+(2m+2)+(2m+1)+[2m-1,2m-1]+\cdots+[3,3]+[1,1].$$

Remark: In this case the weight of the base partition $\beta = 4\binom{m+1}{2} - 2m + 2mn + \binom{n+1}{2}$ and the number of parts in β is $2m+n$.

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General Form of the Forward Moves, Pairs

Suppose we want to push a pair $[b, b]$, there are two cases to consider:

1. The pair becomes $[b+1, b+1]$ and the resulting partition satisfies $rrg_{3,3}$ conditions.
2. We cannot make it $[b+1, b+1]$ because it violates the $rrg_{3,3}$ conditions. Thus, there exists a singleton $(b+2)$ and possibly other singletons $(b+3), (b+4), \cdots, (b+s)$ for some integers $s \geq 2$. Then, forward move on the pair $[b, b]$ is defined as:

$$(b+s) + (b+s-1) + \cdots + (b+3) + (b+2) + [b, b] \to \mathbf{[b+s, b+s]} + (b+s-2) + \cdots + (b+1) + (b)$$

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Suppose we want to push a singleton (a), then (a) becomes $(a+1)$, since there cannot be any problems due to form of the base partition.

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General Form of the Backward Moves, Pairs

Suppose we want to pull a pair $[b+1, b+1]$, there are two cases to consider:

1. The pair becomes, $[b, b]$, and the resulting partition does not violate $rrg_{3,3}$ conditions.
2. We cannot make it $[b, b]$. Thus, there exists a singleton $(b-1)$ and possibly $(b-2), (b-3), \cdots, (b-s)$ where s is an integer $2 \le s < b$. Then, we define the backward move on $[b+1, b+1]$ as:

$$[b+1, b+1] + (b-1) + (b-2) + \cdots + (b-s+1) + (b-s) \to (b+1) + (b) + \cdots + (b-s+2) + \textbf{[b-s,b-s]}$$

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The backward moves on the singletons are straightforward: $(a+1)$ becomes (a), since we pull the singletons first there cannot be any obstacles along the way!

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This directly follows from the order of the moves and "local invertibility".

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Recall

Theorem (Main Theorem)

1. $$\sum_{n,m\geq 0} rrg_{3,1}(m,n)q^n x^m = \sum_{m,n\geq 0} \frac{q^{4\binom{m+1}{2}+2mn+\binom{n+1}{2}+n} x^{2m+n}}{(q^2;q^2)_m (q;q)_n}$$

2. $$\sum_{n,m\geq 0} rrg_{3,2}(m,n)q^n x^m = \sum_{m,n\geq 0} \frac{q^{4\binom{m+1}{2}+2mn+\binom{n+1}{2}} x^{2m+n}}{(q^2;q^2)_m (q;q)_n}$$

3. $$\sum_{n,m\geq 0} rrg_{3,3}(m,n)q^n x^m = \sum_{m,n\geq 0} \frac{q^{4\binom{m+1}{2}+2mn+\binom{n+1}{2}-2m} x^{2m+n}}{(q^2;q^2)_m (q;q)_n}$$

The case $a = 1$

When $a = 1$, everything is same except for the base partition.
In this case the general form of the base partition becomes:

$$(2m+n+1)+(2m+n)+\cdots+(2m+3)+(2m+2)+[2m,2m]+\cdots+[4,4]+[2,2]$$

Remark: Thus, we just push each part of the base partition for the case $a = 3$, 1 unit.

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When $a = 2$, the form of the base partition and definition of the moves change.

Example

When $a = 2$, i.e we can use at most one 1, the smallest weight partition with 1 pair and 1 singletons is neither $[1, 1] + (3)$ nor $[2, 2] + (4)$. It is $(1) + [3, 3]$.

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$a = 2$(Continued)

When $a = 2$ the general form of the base partition is

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In this case the moves on the pairs become straightforward and the moves on the singletons become litte bit trickier.

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Discussion

Let's look at some future ideas:

1. What happens if $k = 4$? Then, we will have three types of parts: singletons, pairs and triples. Thus, we need to look at more cases.. As a rule of thumb, as number of types of parts increase, the combinatorial interpretation in this framework becomes more and more difficult.
2. Best case scenario, we will get a series of the form:

$$\sum_{m,n\geq 0} rrg_{k,k}(m,n)q^n x^m = \sum_{n_1,n_2,\cdots,n_{k-1}\geq 0} \frac{q^{QUADRATIC+LINEAR} x^{LINEAR}}{(q;q)_{n_1}(q^2;q^2)_{n_2}\cdots(q^{n-1};q^{n-1})_{n_{k-1}}}$$

3. Our ultimate goal is to **automatically** interpret any series of the form

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The End

Thank you for your attention!