N-colored generalized Frobenius partitions: Generalized Kolitsch identities

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Table of Contents

Part 1 – Introduction

Partition function

Definition (Partition)

A partition of $n \in \mathbb{N}$ is a non-increasing sequence of positive integers that sums up to be n.

Example

$$14 = 4 + 4 + 4 + 2.$$

Partition function

Definition (Partition function)

The number of different partitions of $n \in \mathbb{N}$ is the partition function denoted by $P(n)$. We define $P(0) = 1$.

Example – $P(4)$

All partitions of 4 are as follows:

$$
\begin{aligned}
4 &= 4, \\
&= 3 + 1, \\
&= 2 + 2, \\
&= 2 + 1 + 1, \\
&= 1 + 1 + 1 + 1.
\end{aligned}
$$

Therefore, $P(4) = 5$.

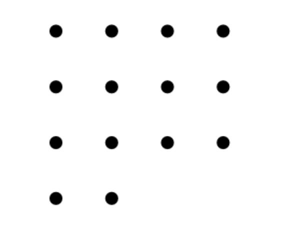
Frobenius symbols

From the Ferrers diagram of a partition, we can construct a 2 by d matrix by carrying out the following steps:

- Remove all the dots lying on the diagonal of the diagram.
- Fill the first row of the matrix with entries $r_{1,j}$, where $r_{1,j}$ is the number of dots on the j-th row that are to the right of the diagonal.
- Fill the second row of the matrix with entries $r_{2,j}$, where $r_{2,j}$ is the number of dots on the j-th column that are below the diagonal.

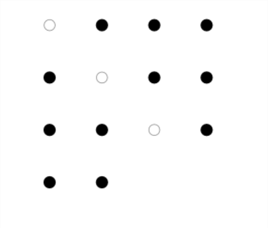
Frobenius symbols - Example (From C-W-Y)

For the partition $14 = 4 + 4 + 4 + 2$ the Ferrer diagram is



Frobenius symbols - Example (From C-W-Y)

For the partition $14 = 4 + 4 + 4 + 2$ the Ferrer diagram is



Therefore the Frobenius symbol for this partition is

$$\begin{pmatrix} 3 & 2 & 1 \\ 3 & 2 & 0 \end{pmatrix}.$$

Generalized Frobenius symbols and Generalized Frobenius partitions

We define the generalized Frobenius symbol, by allowing at most N repetitions in each row of the Frobenius symbol. For a generalized Frobenius symbol with entries $r_{i,j}$, where $i = 1, 2$, and $1 \leq j \leq d$, the generalized Frobenius partition of n is given by

$$n = d + \sum_{j=1}^{d} (r_{1,j} + r_{2,j}).$$

The number of Generalized Frobenius partitions of n is denoted by $\phi_N(n)$.

N-Colored Generalized Frobenius Partitions:

The entries in each row are distinct and are taken from N copies of the non-negative integers distinguished by color and in each row the entries are ordered according to the rule that $x_i < y_j$ if $x < y$ or if $x = y$ and $i < j$ where i and j are integers in the interval $[1, N]$ indicating the color of the non-negative integer. The number of N-colored generalized Frobenius partitions of n is denoted by $c\phi_N(n)$. We note that $c\phi_1(n) = P(n)$, the number of ordinary partitions of n.

Example (From C-W-Y)

Below we list the 2-colored generalized Frobenius symbols which give rise to the 2-colored generalized Frobenius partitions of 2:

$$\begin{pmatrix} 1_1 \\ 0_1 \end{pmatrix}, \begin{pmatrix} 1_1 \\ 0_2 \end{pmatrix}, \begin{pmatrix} 1_2 \\ 0_1 \end{pmatrix}, \begin{pmatrix} 1_2 \\ 0_2 \end{pmatrix},$$
$$\begin{pmatrix} 0_1 \\ 1_1 \end{pmatrix}, \begin{pmatrix} 0_2 \\ 1_1 \end{pmatrix}, \begin{pmatrix} 0_1 \\ 1_2 \end{pmatrix}, \begin{pmatrix} 0_2 \\ 1_2 \end{pmatrix},$$
$$\begin{pmatrix} 0_2 & 0_1 \\ 0_2 & 0_1 \end{pmatrix}.$$

Therefore, we have $c\phi_2(2) = 9$.

Comparison with the partition function

Theorem (Kolitsch (1991))

For all $n \in \mathbb{N}_0$ we have

$$c\phi_5(n) = 5P(5n-1) + P(n/5),$$

$$c\phi_7(n) = 7P(7n-2) + P(n/7),$$

and

$$c\phi_{11}(n) = 11P(11n-5) + P(n/11).$$

Theorem (Chan-Wang-Yang (2019))

For all $n \in \mathbb{N}_0$, we have

$$c\phi_{13}(n) = 13P(13n-7) + P(n/13) + a(n),$$

where $q\dfrac{(q^{13};q^{13})_\infty}{(q;q)_\infty^2} = \sum_{n=1}^{\infty} a(n)q^n$. *When $p \geq 17$ is a prime then we have*

$$\sum_{n=0}^{\infty}\left(c\phi_p(n) - p \cdot P\left(pn - \frac{p^2-1}{24}\right) - P(n/p)\right)q^n = \frac{h_p(z) + 2p^{(p-11)/2}(\eta(pz)/\eta(z))^{p-11}}{(q^p;q^p)_\infty},$$

where $h_p(z)$ is a modular function on $\Gamma_0(p)$ with a zero at ∞ and a pole of order $(p+1)(p-13)/24$ at 0.

Our Results

Theorem (A., Nguyen (2023))

Let N be a squarefree positive integer with $\gcd(N,6)=1$.

i) *Then for all $n \in \mathbb{N}_0$ we have*

$$c\phi_N(n) = \sum_{d|N} \frac{N}{d} \cdot P\left(\frac{N}{d^2}n - \frac{N^2-d^2}{24d^2}\right) + b(n),$$

where $C(z) := (q;q)_\infty^N \sum_{n=1}^\infty b(n)q^n$ is a cusp form in $S_{(N-1)/2}(\Gamma_0(N), \chi_N)$.

ii) *We have $C(z)=0$ if and only if $N = 5, 7,$ or 11.*

iii) *If $N \neq 5, 7,$ or 11, then there is no $M \geq 0$ such that $b(n) = 0$ for all $n > M$.*

Our Results

Theorem (A., Nguyen (2023))

Let N be a squarefree positive integer with $(N, 6) = 1$. We have

$$c\phi_N(n) \sim \sum_{d|N} \frac{N}{d} \cdot P\left(\frac{N}{d^2}n - \frac{N^2 - d^2}{24d^2}\right)$$

as $n \to \infty$.

The Generating Function

Let us denote the generating function of $c\phi_N(n)$ by

$$C\Phi_N(q) := \sum_{n=0}^{\infty} c\phi_N(n) q^n.$$

Andrews has given $C\Phi_N(q)$ in terms of a theta function.

The Generating Function

Let

$$\theta_N(x) := \sum_{i=1}^{N} x_i^2 + \sum_{1 \le i < j \le N} x_i x_j.$$

be a quadratic form in N variables, and

$$f_{\theta_N}(z) := \sum_{x \in \mathbb{Z}^N} q^{\theta_N(x)},$$

be the associated theta function. Then, we have

$$C\Phi_N(z) = \frac{f_{\theta_{N-1}}(z)}{\prod_{n \ge 1} (1 - q^n)^N}.$$

Part 2 – Quadratic forms and Modular forms

Theorem

Let θ be a positive definite quadratic form in $2k$ variables. Then we have

$$f_\theta(z) \in M_k(\Gamma_0(N), \chi_D),$$

where N is the smallest positive integer such that the matrix $N \times Q^{-1}$ has even diagonal entries, where Q denotes the matrix associated with θ and

$$D := \begin{cases} (-1)^k S & \text{if } S \text{ is odd and } (-1)^k S \equiv 1 \pmod 4, \\ (-1)^k 4S & \text{otherwise,} \end{cases}$$

where S denotes the squarefree part of $\det(Q)$.

Matrix associated with $f_{\theta_{N-1}}(z)$

The matrix Q associated with θ_{N-1} is

$$Q = \begin{pmatrix} 2 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & & \\ 1 & 1 & 1 & 1 & \cdots & 2 \end{pmatrix}.$$

It is calculated by Chan, Wang, Yang (2019) that $\det(Q) = N$, and

$$Q^{-1} = \frac{1}{N} \begin{pmatrix} N-1 & -1 & -1 & -1 & \cdots & -1 \\ -1 & N-1 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & & \\ -1 & -1 & -1 & -1 & \cdots & N-1 \end{pmatrix}.$$

Modularity of $f_{\theta_{N-1}}(z)$

Chan-Wang-Yang (2019)

Let N be a squarefree integer with $\gcd(N,6)=1$. We have $f_{\theta_{N-1}}(z) \in M_{(N-1)/2}(\Gamma_0(N), \chi_N)$, where

$$\chi_N(a) = \left(\frac{(-1)^{(N-1)/2}N}{a}\right).$$

Modular forms

The modular subgroup of level $N \in \mathbb{N}$ is defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbb{Z},\ c \equiv 0 \pmod{n},\ ad - bc = 1 \right\}.$$

Let $k \in \mathbb{N}$ and let χ be a Dirichlet character mod N, where $\chi(-1) = (-1)^k$. We denote the space of modular forms of weight k and character χ on $\Gamma_0(N)$ by $M_k(\Gamma_0(N), \chi)$.

The Eisenstein and cusp form subspaces of $M_k(\Gamma_0(N), \chi)$ are denoted by $E_k(\Gamma_0(N), \chi)$ and $S_k(\Gamma_0(N), \chi)$, respectively. Then we have

$$M_k(\Gamma_0(N), \chi) = E_k(\Gamma_0(N), \chi) \oplus S_k(\Gamma_0(N), \chi).$$

Modular forms

Thus, for $f(z) = \sum_{n=0}^{\infty} a_f(n)q^n \in M_k(\Gamma_0(N), \chi)$, there are unique functions

$$E_f(z) = \sum_{n=0}^{\infty} e_f(n)q^n \in E_k(\Gamma_0(N), \chi)$$

and

$$C_f(z) = \sum_{n=0}^{\infty} c_f(n)q^n \in S_k(\Gamma_0(N), \chi),$$

such that

$$a_f(n) = e_f(n) + c_f(n).$$

On the other hand, it is known that

$$c_f(n) = O(n^{k/2}).$$

Next, we describe how we write $e_f(n)$ explicitly in terms of generalized divisor functions defined by

$$\sigma_k(\epsilon, \psi; n) := \begin{cases} \sum_{1 \le d \mid n} \epsilon(n/d)\overline{\psi(d)} d^k & \text{if } n \in \mathbb{N}, \\ 0 & \text{if } n \notin \mathbb{N}, \end{cases}$$

where ϵ and ψ are primitive Dirichlet characters of conductors L and M.

For example if n is odd, we have

$$\begin{aligned} \sigma_k(\chi_{-4}, \chi_1; 2^j \cdot n) &= \left(\sum_{d \mid 2^j} \chi_{-4}(2^j/d)\chi_1(d) d^k \right) \sigma_k(\chi_{-4}, \chi_1; n) \\ &= 2^{jk} \sigma_k(\chi_{-4}, \chi_1; n). \end{aligned}$$

Eisenstein series

We define the weight k Eisenstein series associated with ϵ and ψ by

$$E_k(z;\epsilon,\psi) := \epsilon(0) - \frac{2k}{B_{k,\chi}} \sum_{n=1}^{\infty} \sigma_{k-1}(\epsilon,\psi;n) \mathrm{e}^{2\pi \mathrm{i} n z},$$

where χ is a primitive Dirichlet character such that $\epsilon \cdot \psi = \chi$, and $B_{k,\chi}$ is the Bernoulli number associated with the Dirichlet character χ.

Eisenstein series

The space $E_k(\Gamma_0(N), \chi)$ admits a natural basis of weight k Eisenstein series:

It is known that when $k \geq 2$ and $(k, \chi) \neq (2, \chi_1)$ the collection

$$\mathcal{E}_k(\Gamma_0(N), \chi) = \{E_k(dz; \epsilon, \psi) \mid \epsilon \cdot \psi = \chi \text{ and } LMd \mid N\}$$

forms a basis for the space $E_k(\Gamma_0(N), \chi)$, and when $k = 1$ or $(k, \chi) = (2, \chi_1)$ the collection

$$\mathcal{E}_2(\Gamma_0(N), \chi_1) = \{E_k(dz; \epsilon, \psi) \mid \epsilon \cdot \psi = \chi \text{ and } LMd \mid N\}$$

includes a basis for the space $E_2(\Gamma_0(N), \chi_1)$.

Recall that

Thus, for $f(z)=\sum_{n=0}^{\infty} a_f(n)q^n \in M_k(\Gamma_0(N),\chi)$, there are unique functions

$$E_f(z)=\sum_{n=0}^{\infty} e_f(n)q^n \in E_k(\Gamma_0(N),\chi)$$

and

$$C_f(z)=\sum_{n=0}^{\infty} c_f(n)q^n \in S_k(\Gamma_0(N),\chi),$$

such that

$$a_f(n)=e_f(n)+c_f(n).$$

On the other hand, it is known that

$$c_f(n)=O(n^{k/2}).$$

Previous Results

Theorem (A., 2022-2023)

Let $f(z) \in M_k(\Gamma_0(N), \chi)$, where $N, k \in \mathbb{N}$, $k > 1$, χ is a primitive Dirichlet character with conductor dividing N and satisfying $\chi(-1) = (-1)^k$. Let $E_f(z)$ be the Eisenstein part of f, then

$$E_f(z) = \sum_{(\epsilon,\psi)\in\mathcal{E}(k,N,\chi)} \sum_{d|N/LM} a_f(\epsilon, \psi, d) E_k^*(Mdz; \epsilon, \psi),$$

where

$$a_f(\epsilon, \psi, d) = \prod_{p|N} \frac{p^k}{p^k - \epsilon(p)\psi(p)} \sum_{\substack{c|N/L,\\ M|c}} \mathcal{R}_{k,\epsilon,\psi}(d, c/M)\mathcal{S}_{k,N/LM,\epsilon,\psi}(d, c/M)[f]_{c,\psi},$$

with

$$\mathcal{R}_{k,\epsilon,\psi}(d, c) := \epsilon\left(\frac{-d}{\gcd(d, c)}\right)\psi\left(\frac{c}{\gcd(d, c)}\right)\left(\frac{\gcd(d, c)}{c}\right)^k,$$

$$\mathcal{S}_{k,N,\epsilon,\psi}(d, c) := \mu\left(\frac{dc}{\gcd(d, c)^2}\right) \prod_{\substack{p|\gcd(d,c),\\ 0<v_p(d)=v_p(c)<v_p(N)}} \left(\frac{p^k + \epsilon(p)\psi(p)}{p^k}\right).$$

Corollary

Let N be a squarefree integer with $\gcd(N,6)=1$. Let $f(z)$ be a modular form in $M_{(N-1)/2}(\Gamma_0(N),\chi_N)$. Then we have

$$\begin{aligned} f(z) = {} & [f]_{1/N} \\ & + \sum_{d|N} \frac{[f]_{1/d}}{A(d,N)} \cdot \frac{(1-N)(N/d)^{(N-2)/2}}{B_{(N-1)/2,\chi_N}} \sum_{n\geq 1} \sigma_{(N-3)/2}(\chi_{N/d},\chi_d;n)q^n \\ & + C(z), \end{aligned}$$

where $C(z)$ is some cusp form in $S_{(N-1)/2}(\Gamma_0(N),\chi_N)$ and

$$A(d,N) = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4 \text{ and } N \equiv 1 \pmod 4, \\ \mathrm{i} & \text{if } d \equiv 3 \pmod 4 \text{ and } N \equiv 1 \pmod 4, \\ -\mathrm{i} & \text{if } d \equiv 1 \pmod 4 \text{ and } N \equiv 3 \pmod 4, \\ 1 & \text{if } d \equiv 3 \pmod 4 \text{ and } N \equiv 3 \pmod 4. \end{cases}$$

Part 3 – Modular identities

Constant terms

Let $a \in \mathbb{Z}$ and $c \in \mathbb{N}_0$ be coprime. For an $f(z) \in M_k(\Gamma_0(N), \chi)$ we denote the constant term of $f(z)$ in the Fourier expansion of $f(z)$ at the cusp a/c by

$$[f]_{a/c} := \lim_{z \to \mathrm{i}\infty} (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right),$$

where $b, d \in \mathbb{Z}$ are such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$. The value of $[f]_{a/c}$ does not depend on the choice of b, d.

Modular identities

To get necessary modular identities from the above theorem we need to compute $[f_{\theta_{N-1}}]_{1/d}$ for each $d \mid N$. It is known that we have

$$[f_{\theta_{N-1}}]_{1/d} = \left(\frac{-\mathrm{i}}{d}\right)^{(N-1)/2} \frac{G_{N-1}(1,d)}{\sqrt{N}},$$

where the quadratic Gauss sum $G_N(a,c)$ for $N, a, c \in \mathbb{N}$ is defined by

$$G_N(a,c) := \sum_{x \in (\mathbb{Z}/c\mathbb{Z})^N} \mathrm{e}^{2\pi \mathrm{i} a \theta_N(x)/c}.$$

Therefore to calculate $[f_{\theta_{N-1}}]_{1/d}$ we need to calculate $G_{N-1}(1,d)$.

Gauss sum

Lemma

Let $N \in \mathbb{N}$. Let $\alpha, \beta, \gamma \in \mathbb{N}$ be mutually coprime. Then we have

$$G_N(\gamma, \alpha\beta) = G_N(\beta\gamma, \alpha) G_N(\alpha\gamma, \beta).$$

Proposition

Let p be an odd prime, $N \in \mathbb{N}$ be such that $N \geq p-1$ and $a \in \mathbb{N}$ are coprime to p. Then we have

$$G_N(a,p) = \begin{cases} \mathrm{i}^{(p-p^2)/2} \cdot \left(\dfrac{a}{p}\right) p^{p/2} & \text{if } N = p-1, \text{ or } p, \\ \mathrm{i}^{(p-p^2)/2} \cdot \left(\dfrac{a}{p}\right) p^{p/2} G_{N-p}(a,p) & \text{if } N > p. \end{cases}$$

Gauss sum

Proposition

Let $N > 1$ be an odd positive squarefree integer and let p be a prime divisor of N. If $\gcd(a, p) = 1$, then we have

$$G_{N-1}(a, p) = \mathrm{i}^{(N-Np)/2} \cdot \left(\frac{a}{p}\right) p^{N/2}.$$

Theorem

Let N be an odd positive squarefree integer, let d be a divisor of N, and let $a \in \mathbb{Z}$ with $\gcd(a, d) = 1$. Then we have

$$G_{N-1}(a, d) = \left(\frac{a}{d}\right) \cdot \mathrm{i}^{(N-Nd)/2} \cdot d^{N/2}.$$

Constant term of the theta function

Theorem

Let N be a positive squarefree integer such that $\gcd(N, 6) = 1$ *and d be a divisor of N. Then we have*

$$[f_{\theta_{N-1}}(z)]_{1/d} = \mathrm{i}^{(1-Nd)/2} \cdot \sqrt{d/N}.$$

Theta function in terms of Eisenstein series

Corollary

Let N be a positive squarefree integer such that $\gcd(N,6)=1$. *We have*

$$\begin{aligned} f_{\theta_{N-1}}(z) =& 1+\sum_{d\mid N} C(d,N)(N/d)^{(N-3)/2}\frac{(1-N)}{B_{(N-1)/2,\chi_N}} \\ &\times \sum_{n\geq 1}\sigma_{(N-3)/2}(\chi_{N/d},\chi_d;n)q^n \\ &+ C_3(z), \end{aligned}$$

where $C_3(z)$ is some cusp form in $S_{(N-1)/2}(\Gamma_0(N),\chi_N)$, with

$$C(d,N) := \frac{\mathrm{i}^{(1-Nd)/2}}{A(d,N)} = \left(\frac{-8}{N}\right)\left(\frac{8}{d}\right)\left(\frac{-4}{d}\right)^{(N-1)/2}.$$

Eta quotients

We use eta quotients to relate N-colored Frobenius partitions to the regular partition function.

The Dedekind eta function $\eta(z)$, which is a holomorphic function defined on the upper half plane $\mathbb{H}$ is defined by the product formula

$$\eta(z) = \mathrm{e}^{\pi \mathrm{i} z/12} \prod_{n=1}^{\infty} (1 - \mathrm{e}^{2\pi \mathrm{i} n z}) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) = q^{1/24} (q;q)_\infty.$$

The quotients of products of $\eta(dz)$ for $d \in \mathbb{N}$ in the form

$$\prod_{d|N} \eta^{r_d}(dz), \;\; r_d \in \mathbb{Z},$$

are called **the eta quotients**.

Eta quotients

Generating function of the partition function

We have

$$\frac{q^{1/24}}{\eta(z)} = \sum_{n \geq 0} P(n) q^n.$$

Lemma

Let N be a positive squarefree integer such that $\gcd(N, 6) = 1$. For each $d \mid N$, we have

$$\frac{\eta^N((N/d)z)}{\eta(dz)} \in M_{(N-1)/2}(\Gamma_0(N), \chi_N).$$

Constant terms of eta quotients

Lemma

Let $c \mid N$. Then we have

$$\left[\frac{\eta^N((N/d)z)}{\eta(dz)}\right]_{1/c} = \begin{cases} \left(\frac{N/d}{d}\right) \cdot \left(\frac{d}{N}\right)^{N/2} \cdot \mathrm{i}^{\frac{1-Nd}{2}} & \textit{if } c = d, \\ 0 & \textit{otherwise.} \end{cases}$$

Eta quotients in terms of Eisenstein series

Theorem

Let N be a positive squarefree integer such that $\gcd(N,6)=1$. Then we have

$$\frac{\eta^N((N/d)z)}{\eta(dz)} = \chi_{N/d}(0) + \left(\frac{N/d}{d}\right) C(d,N)\cdot\frac{d}{N}\cdot\frac{(1-N)}{B_{(N-1)/2,\chi_N}} \times \sum_{n\geq 1}\sigma_{(N-3)/2}(\chi_{N/d},\chi_d;n)q^n + C_1(z),$$

where $C_1(z)\in S_{(N-1)/2}(\Gamma_0(N),\chi_N)$.

Partition function in terms of Eisenstein series

For $m \in \mathbb{N}$ we define the operator $U(m)$ by

$$U(m)\Big| \sum_{n\geq 0} a_n q^n = \sum_{n\geq 0} a_{nm} q^n.$$

Recall that

$$\frac{q^{1/24}}{\eta(z)} = \sum_{n\geq 0} P(n) q^n.$$

Applying the operator $U(N/d)$ to the left-hand side of the previous modular identity we obtain:

$$U(N/d)\Big|\frac{\eta^N((N/d)z)}{\eta(dz)} = (q;q)_\infty^N \sum_{n\geq 0} P\left(\frac{N}{d^2} n - \frac{N^2 - d^2}{24 d^2}\right) q^n.$$

If we apply the same operator to the right-hand side of the previous modular identity and use properties of the sum of divisor functions we obtain the next modular identity.

Partition function in terms of Eisenstein series

Theorem

Let N be a positive squarefree integer such that $\gcd(N,6)=1$. Then we have

$$\chi_{N/d}(0) + C(d,N)\cdot (N/d)^{(N-3)/2}\frac{(1-N)}{B_{(N-1)/2,\chi_N}} \times \sum_{n\geq 1}\sigma_{(N-3)/2}(\chi_{N/d},\chi_d;n)q^n$$

$$= N/d\cdot (q;q)_\infty^N \cdot \sum_{n\geq 0} P\left(\frac{N}{d^2}n - \frac{N^2-d^2}{24d^2}\right)q^n + C_2(z),$$

where $C_2(z)$ is some cusp form in $S_{(N-1)/2}(\Gamma_0(N),\chi_N)$.

Recall: Theta function in terms of Eisenstein series

Theorem

Let N be a positive squarefree integer such that $\gcd(N,6)=1$. We have

$$\begin{aligned} f_{\theta_{N-1}}(z) =& 1 + \sum_{d|N} C(d,N)(N/d)^{(N-3)/2} \frac{(1-N)}{B_{(N-1)/2,\chi_N}} \\ & \times \sum_{n\geq 1} \sigma_{(N-3)/2}(\chi_{N/d},\chi_d;n)q^n \\ & + C_3(z), \end{aligned}$$

where $C_3(z)$ is some cusp form in $S_{(N-1)/2}(\Gamma_0(N),\chi_N)$.

Recall that

Theorem (A., Nguyen (2023))

Let N be a squarefree positive integer with $\gcd(N, 6) = 1$.

i) *Then for all* $n \in \mathbb{N}_0$ *we have*

$$c\phi_N(n) = \sum_{d|N} N/d \cdot P\left(\frac{N}{d^2}n - \frac{N^2 - d^2}{24d^2}\right) + b(n),$$

where $C(z) := (q;q)_\infty^N \sum_{n=1}^\infty b(n)q^n$ *is a cusp form in* $S_{(N-1)/2}(\Gamma_0(N), \chi_N)$.

ii) *We have* $C(z) = 0$ *if and only if* $N = 5, 7,$ *or* 11.

iii) *If* $N \neq 5, 7,$ *or* 11, *then there is no* $M \geq 0$ *such that* $b(n) = 0$ *for all* $n > M$.

Proof

We start by proving part i). By combining the previous modular identities we obtain

$$f_{\theta_{N-1}}(z) = (q;q)_\infty^N \sum_{d|N} \frac{N}{d} \sum_{n\geq 0} P\left(\frac{N}{d^2}n - \frac{N^2-d^2}{24d^2}\right) q^n + C(z)$$

for some $C(z) \in S_{(N-1)/2}(\Gamma_0(N), \chi_N)$. We divide both sides of this by $(q;q)_\infty^N$ to obtain

$$\sum_{n\geq 0} c\phi_N(n) q^n = \sum_{n\geq 0} \left(\sum_{d|N} N/d \cdot P\left(\frac{N}{d^2}n - \frac{N^2-d^2}{24d^2}\right)\right) q^n + \frac{C(z)}{(q;q)_\infty^N}.$$

Then the result follows by comparing coefficients of q^n above.

Proof

Now we prove part ii) of the theorem. When $N \geq 29$ a squarefree positive integer coprime to 6 and $d < N$ a divisor of N then $\frac{N}{d^2} - \frac{N^2-d^2}{24d^2} \leq 0$. Therefore since $c\phi_N(1) = N^2$ we have

$$\begin{aligned} b(1) &= c\phi_N(1) - \sum_{d|N} N/d \cdot P\left(\frac{N}{d^2} - \frac{N^2-d^2}{24d^2}\right) \\ &= c\phi_N(1) - P\left(\frac{1}{N}\right) = N^2 \neq 0. \end{aligned}$$

Hence, when $N \geq 29$ is a squarefree positive integer coprime to 6, we have $C(z) \neq 0$.

Proof

Similarly when $N = 13, 17, 19$, or $N = 23$ we have

$$b(1) = c\phi_N(1) - N \cdot P\left(N - \frac{N^2-1}{24}\right) - P\left(\frac{1}{N}\right)$$

$$= \begin{cases} 26 \neq 0 & \text{if } N = 13, \\ 170 \neq 0 & \text{if } N = 17, \\ 266 \neq 0 & \text{if } N = 19, \\ 506 \neq 0 & \text{if } N = 23. \end{cases}$$

This shows that $C(z) \neq 0$ when $N = 13, 17, 19$, or $N = 23$. Therefore by Kolitsch identities, we have $C(z) = 0$ if and only if $N = 5, 7$, or 11.

Proof

Finally, we prove part iii) of the theorem. We prove it by contradiction. Assume that there is an $M \geq 0$ such that $b(n) = 0$ for all $n > M$, then we would have

$$\sum_{n=1}^{\infty} b_n q^n = \sum_{n=1}^{M} b_n q^n = \frac{C(z)}{(q;q)_\infty^N}.$$

The right-hand side of this equation is a meromorphic modular function and the left-hand side is an exponential sum. This is possible only if $\frac{C(z)}{(q;q)_\infty^N} = 0$, which is shown to be false unless $N = 5, 7$, or 11 in the proof of part ii) of the theorem.

Part 4 – Asymptotic behavior

Asymptotic behavior

Let

$$\mathcal{U}(n) = \frac{1-N}{B_{(N-1)/2,\chi_N}} \sum_{d \mid N} C(d,N)(N/d)^{(N-3)/2} \sigma_{(N-3)/2}(\chi_{N/d}, \chi_d; n).$$

We start by investigating the size of $\mathcal{U}(n)$.

Lemma

We have $\mathcal{U}(n) > 0$ *for every* $n \in \mathbb{N}$ *and*

$$\mathcal{U}(n) \gg n^{(N-3)/2} \text{ if } N > 5,$$
$$\mathcal{U}(n) \gg n/\log\log n \text{ if } N = 5.$$

For each non-negative integer r, we define $\mathcal{V}_r(n)$ for $n \geq 0$ by:

$$\sum_{n\geq 0} \mathcal{V}_r(n) q^n = \frac{1}{(q;q)_\infty^r} = \left(\sum_{n\geq 0} P(n) q^n\right)^r = \sum_{n\geq 0} \sum_{\substack{x\in\mathbb{N}_0^r \\ \sum x_i = n}} \prod_{i=1}^r P(x_i) q^n.$$

We have:

Proposition

For $r \geq 1$:

(i) $\displaystyle \lim_{n\to\infty} \frac{\mathcal{V}_r(n)}{\mathcal{V}_r(n-1)} = 1.$

(ii) $\displaystyle \lim_{n\to\infty} \frac{\mathcal{V}_{r-1}(n)}{\mathcal{V}_r(n)} = 0.$

Proof – Asymptotic behavior

When $N = 5$, 7 or 11 from Sturm's theorem, we have $c\phi_N(n) = \sum_{d|N} N/d \cdot P\left(\frac{N}{d^2}n - \frac{N^2-d^2}{24d^2}\right) (\neq 0)$. Therefore the statement for $N = 5$, 7 or 11 follows immediately. From now on assume $N > 11$. By a modular identity from before we have

$$f_{\theta_{N-1}}(z) - 1 - \sum_{n\geq 1} \mathcal{U}(n) q^n \in S_{(N-1)/2}(\Gamma_0(N), \chi_N).$$

Proof – Asymptotic behavior

Thus, by using Hecke bound, we have

$$f_{\theta_{N-1}}(z) - 1 - \sum_{n\geq 1} \mathcal{U}(n) q^n = \sum_{n \geq 1} O(n^{(N-1)/4}) q^n.$$

On the other hand, by another modular identity, we have

$$(q;q)_\infty^N \sum_{d|N} (N/d) \sum_{n\geq 0} P\left(\frac{N}{d^2} n - \frac{N^2-d^2}{24d^2}\right) q^n - 1 - \sum_{n\geq 1} \mathcal{U}(n) q^n \in S_{(N-1)/2}(\Gamma_0(N), \chi_N).$$

Hence, by using Hecke bound, we obtain

$$(q;q)_\infty^N \sum_{d|N} (N/d) \sum_{n\geq 0} P\left(\frac{N}{d^2} n - \frac{N^2-d^2}{24d^2}\right) q^n - 1 - \sum_{n\geq 1} \mathcal{U}(n) q^n$$
$$= \sum_{n\geq 1} O(n^{(N-1)/4}) q^n.$$

Proof – Asymptotic behavior

Now we let $\mathcal{V}(n) := \mathcal{V}_N(n)$ so that

$$\frac{1}{(q;q)_\infty^N} = \sum_{n\geq 0} \mathcal{V}(n) q^n.$$

With this notation and the earlier arguments, we obtain

$$c\phi_N(n) - \sum_{\ell+m=n} \mathcal{V}(m)\mathcal{U}(l) = O\left(\sum_{\ell+m=n} \mathcal{V}(m)\ell^{(N-1)/4}\right),$$

and

$$\sum_{d|N}(N/d)\sum_{n\geq 0} P\left(\frac{N}{d^2}n - \frac{N^2-d^2}{24d^2}\right) - \sum_{\ell+m=n} \mathcal{V}(m)\mathcal{U}(\ell)$$
$$= O\left(\sum_{\ell+m=n} \mathcal{V}(m)\ell^{(N-1)/4}\right).$$

Proof – Asymptotic behavior

From above we have

$$\lim_{n\to\infty} \frac{c\phi_N(n)}{\sum_{d|N}(N/d)P\left(\frac{N}{d^2}n - \frac{N^2-d^2}{24d^2}\right)}$$

$$= \lim_{n\to\infty} \frac{\sum_{\ell+m=n} \mathcal{V}(m)\mathcal{U}(\ell) + O\left(\sum_{\ell+m=n} \mathcal{V}(m)\ell^{(N-1)/4}\right)}{\sum_{\ell+m=n} \mathcal{V}(m)\mathcal{U}(\ell) + O\left(\sum_{\ell+m=n} \mathcal{V}(m)\ell^{(N-1)/4}\right)}.$$

Proof – Asymptotic behavior

To obtain the desired result, we prove:

$$\sum_{\ell+m=n} \mathcal{V}(m)\ell^{(N-1)/4} = o\left(\sum_{\ell+m=n} \mathcal{V}(m)\mathcal{U}(\ell)\right) \text{ as } n \to \infty.$$

Since $N > 11$, we have that $\mathcal{U}(\ell) \gg \ell^{(N-3)/2}$ dominates $\ell^{(N-1)/4}$ when ℓ is large.

Result

Theorem (A., Nguyen (2023))

Let N be a squarefree positive integer with $(N, 6) = 1$. We have

$$c\phi_N(n) \sim \sum_{d|N} N/d \cdot P\left(\frac{N}{d^2}n - \frac{N^2 - d^2}{24d^2}\right)$$

as $n \to \infty$.

Thanks!