Finding odd values of $p(n)$

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The Partition Function

Definition

$$p(n) := \#\left\{\lambda = (\lambda_1 \geq \ldots \geq \lambda_r > 0) \middle| r > 0, \sum_i \lambda_i = n\right\}.$$

Example

$p(4) = 5$ because

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

Hardy-Ramanujan-Rademacher Formulas

Theorem (Hardy-Ramanujan, 1917)

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$$

Theorem (Rademacher)

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k} \frac{d}{dn} \left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right)}{\sqrt{n - \frac{1}{24}}} \right)$$

where

$$A_k(n) := \sum_{\substack{0 \le h < k \\ (h,k)=1}} e^{\pi i s(h,k) - 2\pi i n h/k} \text{ with } s(h,k) = \sum_{r=1}^{k-1} \frac{r}{k} \left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2} \right)$$

Partition Congruences

Theorem (Ramanujan, 1919)

For all $m \in \mathbb{N}$, we have

$$p(5m+4) \equiv 0 \pmod 5$$

$$p(7m+5) \equiv 0 \pmod 7$$

$$p(11m+6) \equiv 0 \pmod{11}$$

Theorem (Watson, 1938; Atkin, 1967)

$$p(n) \equiv 0 \pmod{5^a} \text{ if } 24n \equiv 1 \pmod{5^a}$$
$$p(n) \equiv 0 \pmod{7^b} \text{ if } 24n \equiv 1 \pmod{7^{2b-2}}$$
$$p(n) \equiv 0 \pmod{11^c} \text{ if } 24n \equiv 1 \pmod{11^c}$$

Partition Congruences

Theorem (Atkin, 1968; Atkin-O'Brien, 1967)

$$p(11^3 \cdot 13m + 237) \equiv 0 \pmod{13}$$

$$p(169n - 7) \equiv \kappa_d p(n) \pmod{13^d} \text{ if } 24n \equiv 1 \pmod{13^d}$$

where κ_d is an integer depending only on d.

Theorem (Ono, 2000)

Let $m \geq 5$ be a prime, and let k be a positive integer. A positive proportion of the primes ℓ have property that

$$p\left(\frac{m^k \ell^3 n + 1}{24}\right) \equiv 0 \pmod{m}$$

for every nonnegative integer n coprime to ℓ.

Parity Distributions of $p(n)$

x	$\#\{p(n) \text{ odd} : n \leq x\}$	Proportion
50000	25016	0.50032
100000	50200	0.50200
150000	75041	0.50027
200000	99766	0.49883
250000	124703	0.49881
300000	149758	0.49919

Table: Odd values of $p(n)$

Parity Distributions of $p(n)$

Conjecture (Parkin-Shanks, 1967)

$$\lim_{x\to\infty} \frac{\#\{p(n) \text{ odd} : n \leq x\}}{\#\{p(n) : n \leq x\}} = \frac{1}{2}.$$

Theorem (Bellaïche-Nicolas, 2016)

$$\#\{p(n)\ \ odd : n \leq x\} \gg \sqrt{x}/(\log x)^{7/8}.$$

Subbarao's conjecture

Theorem (Ono, 1996; Radu, 2012)

If $t > r \geq 0$, then

$$\begin{aligned} |\{N \equiv r \pmod{t} : p(N) \ \text{odd}\}| = \infty, \\ |\{M \equiv r \pmod{t} : p(M) \ \text{even}\}| = \infty. \end{aligned}$$

Open Problem

Construct explicit sequences of partition values of known parity.

Finite sets

Theorem (Euler)

$$\prod_{n=1}^{\infty}(1-q^n) = 1 + \sum_{n=1}^{\infty}(-1)^n\left(q^{\frac{n(3n-1)}{2}} + q^{\frac{n(3n+1)}{2}}\right).$$

Theorem (Gauss)

$$\sum_{n=0}^{\infty} q^{\mathcal{T}_n} = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{1-q^n}, \quad \mathcal{T}_n = n(n+1)/2$$

Construction

$$\sum_{n=0}^{\infty} q^{\mathcal{T}_n} \equiv \prod_{n=1}^{\infty}(1-q^{4n}) \cdot \sum_{n=1}^{\infty} p(n)q^n \pmod 2$$

$\Longrightarrow p\left(\mathcal{T}_n - \left(6j^2 \pm 2j\right)\right)$ is odd for some j.

Sets of $p(n)$ with odd elements

Ono's Question

Infinitely many nontrivial finite sets with some odd $p(n)$?

Theorem 1 (C.-Z.)

If $\ell > 0$ is odd, $5 \nmid \ell$, the following finite sets contain an odd element:

$$\left\{ p\left(\frac{5\ell^2 - 5}{8} - 16j \right) : j \in \mathbb{Z}_{\geq 0} \right\} \tag{1}$$

$$\left\{ p\left(\frac{125\ell^2 - 21}{8} - 64j \right) : j \in \mathbb{Z}_{\geq 0} \right\}. \tag{2}$$

Examples

Example

ℓ	Values from Set (1)	Values from Set (2)
1	$\{p(0) = 1\}$	$\{p(13) = 101\}$
3	$\{p(5) = 7\}$	$\{p(10), p(74), p(138)\}$
7	$\{p(14), p(30)\}$	$\{p(59), p(123), \dots, p(763)\}$
9	$\{p(18), p(34), p(50)\}$	$\{p(47), p(111), \dots, p(1263)\}$
11	$\{p(11), p(27), p(43), p(59), p(75)\}$	$\{p(32), p(96), \dots, p(1888)\}$

Table: Odd $p(n)$'s in blue

Remark

The theorem is sharp.

How do we prove this?

Modular forms
mod 2
and
Hecke nilpotency

Modular forms

Definition

$f : \mathcal{H} \to \mathbb{C}$ is a *weight k modular (resp. cusp) form* over $\mathrm{SL}_2(\mathbb{Z})$ if:

1. f is holomorphic;
2. $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$;
3. $f = \sum_{n \geq n_0} a_f(n) q^n$ $(q := e^{2\pi i z})$, $n_0 \geq 0$ (resp. $n_0 \geq 1$).

Definition

M_k (resp. S_k) = space of weight k modular (resp. cusp) forms.

Examples of Modular Forms

Eisenstein series

$$E_4(z) = 1 + 240\sum_{n\geq 1}\sigma_3(n)q^n, \quad E_6(z) = 1 - 504\sum_{n\geq 1}\sigma_5(n)q^n.$$

We have $E_4 \equiv E_6 \equiv 1 \pmod{2}$.

Delta function

$$\Delta(q) := q\prod_{n=1}^{\infty}(1-q^n)^{24}.$$

Remark

The elements of $M_k \cap \mathbb{Z}[[q]]$ are polynomials in E_4, E_6, Δ.

Modular Forms Mod 2

Definition

Let $\widetilde{M}_k = \{\tilde{f} = \sum \tilde{a}_f(n)q^n = f \pmod 2 : f \in M_k \cap \mathbb{Z}[[q]]\}$,
$\widetilde{S}_k = \{\tilde{f} = \sum \tilde{a}_f(n)q^n = f \pmod 2 : f \in S_k \cap \mathbb{Z}[[q]]\}$.

Fact

$\sum \widetilde{M}_k$ can be identified by $\mathbb{F}_2[\Delta] \subset \mathbb{Z}[[q]]$.

Proposition

$$\Delta \equiv \sum_{m=1}^{\infty} q^{(2m+1)^2} \pmod 2.$$

Hecke operators

Notation

The *Hecke operator* T_p for prime p acts by

$$T_p \mid f := \sum_{n=1}^{\infty} \left(a_f(pn) + p^{k-1} a_f(n/p) \right) q^n,$$

where $a_f(n/p) := 0$ if $p \nmid n$.

Fact

T_p preserves M_k (resp. S_k).

Remark

Hecke operators commute: $T_{p_1} T_{p_2} \mid f = T_{p_2} T_{p_1} \mid f$.

Hecke operators mod 2

Action of T_p modulo 2

If $f = \sum a(n)q^n$, we have

$$T_p \mid f \equiv \sum_{n=1}^{\infty}(a(pn) + a(n/p))q^n \pmod 2 \quad (p \neq 2),$$

$$T_2 \mid f \equiv \sum_{n=1}^{\infty} a(2n)q^n \pmod 2.$$

Hecke nilpotency

Theorem (Serre, Tate 1994)

T_p $(p \neq 2)$ act "locally nilpotently" on $\mathbb{F}_2[\Delta]$.

Definition

The *degree of nilpotency* of a cusp form f is

$$d(f) := \min\{d \geq 0 : T_{p_1} \ldots T_{p_d} \mid f = 0,\ \forall\, p_1, \ldots, p_d \text{ primes}\}.$$

Examples

Examples

- $T_p \mid \Delta = 0$ for any prime p, so $d(\Delta) = 1$.
- $d(\Delta^3) = 2$ since $T_p \mid \Delta^3 = \begin{cases} \Delta & \text{if } p \equiv 3 \pmod 8 \\ 0 & \text{otherwise.} \end{cases}$

Question

What about $d(f)$ for $f = \Delta^{128} + \Delta^{60} + \Delta^7 + \Delta^4$?

Remark

In general, $d(f + g) \leq \max\{d(f), d(g)\}$, but not equal.

Natural questions

Questions

1. How does one overcome $d(f+g) \neq \max\{d(f), d(g)\}$?
2. How does one calculate $d(f)$?

Computing $d(f)$

Theorem 2 (C.-Z.)

Let $f = \sum_{m_i > 0} \Delta^{m_i}$. *Write* $f = \sum_j f_j$, *where* $f_j := \sum_{\nu_2(m_i)=j} \Delta^{m_i}$. Then

1. *For each* j, *we have* $d(f_j) = j + d\left(\sum_{\nu_2(m_i)=j} \Delta^{m_i/2^j}\right)$.
2. $d(f) = \max_j \{d(f_j)\}$.

Remark

Theorem 2 is a key part of our algorithm for computing $d(f)$.

Work of Nicolas and Serre

Fact

The Hecke algebra T_p, $p \neq 2$ is determined by T_3 and T_5.

Definition

- The *code* of $m = \sum_{i=0}^{\infty} \beta_i 2^i$ is $[n_3(m), n_5(m)] \in \mathbb{N}^2$, where

$$n_3(k) := \sum_{i=0}^{\infty} \beta_{2i+1} 2^i, \quad n_5(k) := \sum_{i=0}^{\infty} \beta_{2i+2} 2^i.$$

- The *height* of m is $h(m) = n_3(m) + n_5(m)$.

Example

$$13 = 1{\cdot}2^3 + 1{\cdot}2^2 + 0{\cdot}2^1 + 1{\cdot}2^0$$
$$n_3(13) = 1{\cdot}2^1 + 0{\cdot}2^0 = 2, \; n_5(13) = 1{\cdot}2^0 = 1 \Rightarrow h(13) = 2 + 1 = 3$$

Work of Nicolas and Serre

Notation

Let $\mathcal{F} \subset \mathbb{F}_2[\Delta]$ denote the subspace generated by $\Delta, \Delta^3, \Delta^5, \ldots$.
Let $\succ$ order odd m lexicographically by $(h(m), n_5(m))$.

Theorem (Nicolas–Serre, 2012)

For $f = \Delta^{m_1} + \cdots + \Delta^{m_r} \in \mathcal{F}$, with $m_1 \succ m_2 \succ \ldots$, ordered by code,

$$T_3^{n_3(m_1)} T_5^{n_5(m_1)} \mid f = \Delta.$$

Moreover, $d(f) = h(f) + 1 = h(m_1) + 1$.

Example

Example (Odd exponents)

Let $f = \Delta^{17} + \Delta^{15} + \Delta^{7}$. Then, we have

$$[n_3(17), n_5(17)] = [2, 0],$$
$$[n_3(15), n_5(15)] = [3, 1],$$
$$[n_3(7), n_5(7)] = [1, 1].$$

Thus, we have

$$d(f) = h(f) + 1 = h(15) + 1 = n_3(15) + n_5(15) + 1 = 5.$$

Proof of Theorem 2

Recall: Theorem 2

Let $f = \sum_{m_i>0} \Delta^{m_i}$. *Set* $f_j := \sum_{\nu_2(m_i)=j} \Delta^{m_i}$ *and write* $f = \sum_j f_j$.
(1) $d(f_j) = j + d\left(\sum_{\nu_2(m_i)=j} \Delta^{s_i}\right)$, *where* $s_i := m_i/2^{\nu_2(m_i)}$.
(2) $d(f) = \max_j\{d(f_j)\}$.

Proof.

1) $T_2 \mid f \equiv \sum_{n>0} a_f(2n)q^n \pmod 2 \Rightarrow T_2 \mid \sum_{m_i\in\mathbb{N}} \Delta^{2m_i} = \sum_{m_i\in\mathbb{N}} \Delta^{m_i}$.

$T_2^j \mid f_j \equiv \sum \Delta^{s_i} \pmod 2 \implies d(f) \geq d\left(\sum \Delta^{s_i}\right) + j$.

Converse inequality: induction on j.

(2) Goal: $d(f_i + f_j) \geq \max\{d(f_i), d(f_j)\}$. WLOG, $d(f_i) > d(f_j)$.

Indeed, $T_3^{n_3(T_2^i|f_i)} T_5^{n_5(T_2^i|f_i)} T_2^i \mid (f_i + f_j) = \Delta$. □

Algorithm for computing $d(f)$

Let $f = \sum \Delta^{m_i}$ be a cusp form.

Step 1. Set $f_j = \sum_{\nu_2(m_i)=j} \Delta^{m_i}$.

Step 2. For each j, compute

$$d\Big(\sum_{\nu_2(m_i)=j} \Delta^{m_i/2^j} \Big) = \max_{\nu_2(m_i)=j} \{h(m_i/2^j)\} + 1.$$

Step 3. For each j, compute

$$d(f_j) = j + d\Big(\sum_{\nu_2(m_i)=j} \Delta^{m_i/2^j} \Big),$$

Step 4. Find the degree of nilpotency of f using

$$d(f) = \max_{0 \le j \le v} \{d(f_j)\}.$$

Example of computing $d(f)$

Example (Mixed exponents)

Consider $f = \Delta^{128} + \Delta^{60} + \Delta^{7} + \Delta^{4}$.

(1) Decompose:

$$f = \Delta^{128} + (\Delta^{60} + \Delta^{4}) + \Delta^{7} = \Delta^{2^7} + (\Delta^{15\cdot 2^2} + \Delta^{2^2}) + \Delta^{7}.$$

(2) Compute degree of nilpotency for each component:

$$d(\Delta^{128}) = 7 + d(\Delta^{1}) = 8,$$
$$d(\Delta^{60} + \Delta^{4}) = d(\Delta^{15\cdot 2^2} + \Delta^{2^2}) = 2 + d(\Delta^{15}) = 2 + h(15) = 7,$$
$$d(\Delta^{7}) = n_3(7) + n_5(7) + 1 = 3.$$

(3) Assemble:

$$d(f) = \max\{d(\Delta^{128}), 2 + d(\Delta^{15}), d(\Delta^{7})\} = \max\{8, 7, 3\} = 8.$$

Natural Question

Question

How many cusp forms have a given degree of nilpotency?

First Guess

Seems plausible that there are infinitely many!

Counting f with fixed $d(f)$

Theorem 3 (C.-Z.)

If $n \in \mathbb{N}$, *then*

$$\#\{f \in \mathbb{F}_2[\Delta] : d(f) = n\} = 2^{\frac{n(n+1)(n+2)}{6}} - 2^{\frac{(n-1)n(n+1)}{6}}.$$

Proof.

- For $f = \sum_i \Delta^{m_i}$, define $s_i = m_i/2^{\nu_2(m_i)}$. Then

$$d(f) = \max_i\{n_3(s_i) + n_5(s_i) + v_2(m_i) + 1\}.$$

- There are $\binom{n+2}{3}$ triples with

$$n_3(s_i) + n_5(s_i) + \nu_2(m_i) + 1 \leq n.$$

- This gives us $2^{\binom{n+2}{3}}$ cusp forms with $d(f) \leq n$.

Natural Question

Question

- How are degrees of nilpotency distributed in spaces $\widetilde{S}_k$?
- Any limiting distribution as $k \to \infty$?

Distributions

Theorem 4 (C.-Z.)

The degrees of Hecke nilpotency in spaces $\widetilde{S}_k$ do not have a limiting distribution as $k \to \infty$.

Strategy

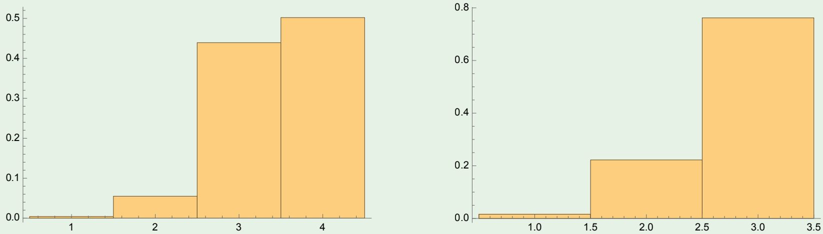
We construct sequences of weights $\{k_n\}, \{k'_n\}$ such that

$$\frac{\#\{f : d(f) = m_{k_n}\}}{|\widetilde{S}_{k_n}|} = \frac{1}{2}, \quad \frac{\#\{f : d(f) = m_{k'_n}\}}{|\widetilde{S}_{k'_n}|} = \frac{3}{4},$$

where $m_k := \max_f \{d(f) : \text{weight of } f \leq k\}$.

Histograms

Example



$$\{k_n\} = \{12, 96, \ldots, 12 + 12\sum_{i=0}^{n-1} 2^{2i+1}, \ldots\} \qquad \{k'_n\} := \{k_n + 24\}$$

Figure: Histograms for weights 96 and 72.

Completing the story

Question

How does Hecke nilpotency relate to the parity of $p(n)$?

Parity of $p(n)$

Theorem 5 (C.-Z.)

Fix $s > 1$ and odd $\ell > 0$ with $5 \nmid \ell$. There exists some

$$i_0 \in \{i \text{ odd} : 0 < i \leq 2^{s-1} - 1, 2^r \neq 2^{s-1} - i - 1, \ \forall\, 0 \leq r \leq s-2\}$$

such that the following finite set contains an odd element:

$$\left\{ p\left(\frac{5^{i_0}\ell^2 - \frac{4^s - 1}{3}}{8} - 4^s \cdot j \right) \,\middle|\, j \in \mathbb{Z}_{\geq 0} \right\}.$$

Remark

- $s = 2 \implies$ Theorem 1, set (1).
- $s = 3 \implies$ Theorem 1, set (2).

Proof of Theorem 5

Parity of $p(n)$

$\updownarrow$ q-series

Parity of coefficients of $\Delta^{(4^s-1)/3}$

$\updownarrow$ combinatorics

Parity of coefficients of $T_5^{2^{s-1}-1}|\Delta^{(4^s-1)/3}$

$\updownarrow$ code

Parity of coefficients of Δ

Step 1: q-series

Let $s > 1$.

$$\Delta^{\frac{4^s-1}{3}} = \left(q\prod_{n=1}^{\infty}(1-q^n)^{24}\right)^{\frac{4^s-1}{3}}$$
$$\equiv \left(1+\sum_{j=1}^{\infty} r_s(j)q^{8\cdot 4^s j}\right)\cdot\left(\sum_{k=0}^{\infty} p(k)q^{8k+\frac{4^s-1}{3}}\right) \pmod 2$$

Using the convolution formula and comparing coefficients:

$$a_{\Delta^{\frac{4^s-1}{3}}}(n) \equiv p\left(\frac{n-\frac{4^s-1}{3}}{8}\right) + \sum_{j=1}^{\infty} r_s(j)p\left(\frac{n-\frac{4^s-1}{3}}{8} - 4^s j\right) \pmod 2.$$

Step 3: Code

- $$\frac{4^s-1}{3} = 1 + 0\cdot 2 + 1\cdot 2^2 + 0\cdot 2^3 + 1\cdot 2^4 + \cdots + 1\cdot 2^{2s-2}.$$
- $$\left[n_3\left(\frac{4^s-1}{3}\right), n_5\left(\frac{4^s-1}{3}\right)\right] = [0, 1+2+\cdots+2^{s-2}] = [0, 2^{s-1}-1].$$
- Using the code, we have,

$$T_5^{2^{s-1}-1} \mid \Delta^{\frac{4^s-1}{3}} \equiv \Delta \pmod 2.$$

Step 2: Combinatorics

Step 2: Combinatorics

- $a_{T_p^t|f}(n)$ is a linear combination of $\{a_f(p^i n) : |i| \leq t, 2 \mid (i-t)\}$.

$$\begin{array}{lccccccccc}
a_f(n): & & & & & a(n) & & & & \\
& & & & / & & \backslash & & & \\
a_{T_p|f}(n): & & & & a(pn) & & a(p^{-1}n) & & & \\
& & & / & & \backslash\ / & & \backslash & & \\
a_{T_p^2|f}(n): & & & a(p^2n) & & 2a(n) & & a(p^{-2}n) & & \\
& & / & & \backslash\ / & & \backslash\ / & & \backslash & \\
a_{T_p^3|f}(n): & & a(p^3n) & & 3a(pn) & & 3a(p^{-1}n) & & a(p^{-3}n) & \\
& / & & \backslash\ / & & \backslash\ / & & \backslash\ / & & \backslash \\
a_{T_p^4|f}(n): & a(p^4n) & & 4a(p^2n) & & 6a(n) & & 4a(p^{-2}n) & & a(p^{-4}n)
\end{array}$$

Warning

$a(n/p^i) := 0$ whenever $p^i \nmid n$.

Step 2 (Cont.)

Corrected diagram

For example, in the case where $\nu_p(n) = 0$, we have:

$a_f(n)$:					$a(n)$				
$a_{T_p\mid f}(n)$:				$a(pn)$		0			
$a_{T_p^2\mid f}(n)$:			$a(p^2n)$		$a(n)$		0		
$a_{T_p^3\mid f}(n)$:		$a(p^3n)$		$2a(pn)$		0		0	
$a_{T_p^4\mid f}(n)$:	$a(p^4n)$		$3a(p^2n)$		$2a(n)$		0		0

Notation

$c_{p,t,i}(n)$ is the coefficient associated with $a(p^in)$ in the tth row.

Step 2 (Cont.)

Lemma

Given $t, n \in \mathbb{N}$, we have

$$c_{p,t,i}(n) = \begin{cases} c_{p,t-1,i-1}(n) + c_{p,t-1,i+1}(n) & \text{if } \max\{-\nu_p(n), -t\} \le i \le t \\ 0 & \text{if } i < \max\{-\nu_p(n), -t\} \text{ or } i > t. \end{cases}$$

Idea

$c_{p,t,i}(n)$ can be described by a modified Pascal's triangle.

$$\boxed{\nu_p(n) = 0}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1

$1 \quad 1\ {-1}$

$1 \quad 2\ {-1} \quad 1\ {-1}$

$1 \quad 3\ {-1} \quad 3\ {-2} \quad 1\ {-1}$

$1 \quad 4\ {-1} \quad 6\ {-3} \quad 4\ {-3} \quad 1\ {-1}$

$1 \quad 5\ {-1} \quad 10\ {-4} \quad 10\ {-6} \quad 5\ {-4} \quad 1\ {-1}$

$1 \quad 6\ {-1} \quad 15\ {-5} \quad 20\ {-10} \quad 15\ {-10} \quad 6\ {-5} \quad 1\ {-1}$

1

1 0

1 1 0

1 2 1 0

1 3 3 1 0

1 4 6 4 1 0

1 5 10 10 5 1 0

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 0 & & & & & \\
 & & & & 1 & & 1 & & 0 & & & & \\
 & & & 1 & & 2 & & 1\ {-1} & & 0 & & & \\
 & & 1 & & 3 & & 3\ {-1} & & 1\ {-1} & & 0 & & \\
 & 1 & & 4 & & 6\ {-1} & & 4\ {-2} & & 1\ {-1} & & 0 & \\
1 & & 5 & & 10\ {-1} & & 10\ {-3} & & 5\ {-3} & & 1\ {-1} & & 0
\end{array}
$$

1

1 0

1 1 0

1 2 0 0

1 3 2 0 0

1 4 5 2 0 0

1 5 9 7 2 0 0

1

$1 \quad 0$

$1 \quad 1 \quad 0$

$1 \quad 2 \quad 0 \quad 0$

$1 \quad 3 \quad 2 \quad 0 \quad 0$

$1 \quad 4 \quad 5 \quad 2 - 2 \cdot 1 \quad 0 \quad 0$

$1 \quad 5 \quad 9 \quad 7 - 2 \cdot 1 \quad 2 - 2 \cdot 1 \quad 0 \quad 0$

1

1 0

1 1 0

1 2 0 0

1 3 2 0 0

1 4 5 0 0 0

1 5 9 5 0 0 0

Step 2 (Cont.)

Proposition

Let $v := \nu_p(n)$.

$$c_{p,t,i}(n) \equiv \binom{t}{\frac{t-i}{2}} - \sum_{1 \le \ell \le \frac{t-1}{2}} \binom{2\ell + v - 1}{\ell + \lceil \frac{v}{2} \rceil} \binom{t - 2\ell + 1 - v}{\frac{t-i}{2} - \ell - \lceil \frac{v}{2} \rceil} \pmod 2.$$

Remark

- $p^t \mid n \Rightarrow \binom{t}{\frac{t-i}{2}}$ is sufficient
- $p^t \nmid n \Rightarrow$ remove overcounted contributions from $a(n/p^m)$, $m > \nu_p(i)$.

Step 2 (Cont.)

Lemma

If $5 \nmid n$, then

$$c_i := c_{5,2^{s-1}-1,i}(n) \equiv \begin{cases} 0 \pmod 2 & \text{if } i = 2^{s-1} - 2^r - 1 \text{ for some } r \in \mathbb{Z} \\ 1 \pmod 2 & \text{if } i \neq 2^{s-1} - 2^r - 1 \ \forall r \in \mathbb{Z}\,. \end{cases}$$

Proof.

$$c_i \equiv \binom{2^{s-1}-1}{\frac{2^{s-1}-1-i}{2}} - \sum_{1 \leq \ell < 2^{s-2}-1} \binom{2\ell-1}{\ell} \binom{2^{s-1}-2\ell}{\frac{2^{s-1}-1-i}{2} - \ell} \pmod 2.$$

$\binom{2^{s-1}-1}{a}$ is odd $\forall\ 0 \leq a \leq 2^{s-1}-1$, and $\binom{2\ell-1}{\ell}$ is odd iff $\ell = 2^r$, so

$$c_i \equiv 1 + \sum_{1 \leq r \leq s-3} \binom{2^{s-1} - 2^{r+1}}{\frac{2^{s-1}-1-i}{2} - 2^r} \pmod 2.$$

Proof of Theorem 5

Proof.

Set $m = \frac{4^s-1}{3}$ and choose odd ℓ with $5 \nmid \ell$.

$$T_5^{2^{s-1}-1} \mid \Delta^m \equiv \Delta \pmod 2 \implies \sum_{\substack{0 \le i \le 2^{s-1}-1 \text{ odd} \\ 2^r \ne 2^{s-1}-i-1\ \forall r \ge 0}} a_{\Delta^m}(5^i \ell^2) \equiv 1 \pmod 2.$$

Since $a_{\Delta^m}(5^{i_0}\ell^2)$ is odd for some i_0, there is some $j \ge 0$ such that

$$p\left(\frac{5^{i_0}\ell^2 - \frac{4^s-1}{3}}{8} - 4^s j\right) \equiv 1 \pmod 2.$$

□

Summary

Theorem (C.-Z.)

$$d(f) = \max_{0 \le j \le v} \left\{ j + d\left(\sum_{\nu_2(m_i)=j} \Delta^{m_j/2^j} \right) \right\}.$$

Theorem (C.-Z.)

If $n > 0$, then $\#\{f : d(f) = n\} = 2^{\frac{n(n+1)(n+2)}{6}} - 2^{\frac{(n-1)n(n+1)}{6}}$.

Theorem (C.-Z.)

Fix $s > 1$ and $\ell > 0$ odd, $5 \nmid \ell$. Then there is some odd element in

$$\left\{ p\left(\frac{5^i \ell^2 - \frac{4^s - 1}{3}}{8} - 4^s \cdot j \right) : \; j \in \mathbb{Z}_{\ge 0} \right\}$$

for some odd $i \le 2^{s-1} - 1$, $i \ne 2^{s-1} - 2^r - 1 \;\forall\; 0 \le r \le s-2$.