Second-Order Consensus Seeking in Multi-Agent Systems With Nonlinear Dynamics Over Random Switching Directed Networks

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***Abstract*—This paper discusses the second-order local consensus problem for multi-agent systems with nonlinear dynamics over dynamically switching random directed networks. By applying the orthogonal decomposition method, the state vector of resulted error dynamical system can be decomposed as two transversal components, one of which evolves along the consensus manifold and the other evolves transversally with the consensus manifold. Several sufficient conditions for reaching almost surely second-order local consensus are derived for the cases of time-delay-free coupling and time-delay coupling, respectively. For the case of time-delay-free coupling, we find that if there exists one directed spanning tree in the network which corresponds to the fixed time-averaged topology and the switching rate of the dynamic network is not more than a critical value which is also estimated analytically, then second-order dynamical consensus can be guaranteed for the choice of suitable parameters. For the case of time-delay coupling, we not only prove that under some assumptions, the second-order consensus can be reached exponentially, but also give an analytical estimation of the upper bounds of convergence rate and the switching rate. Finally, numerical simulations are provided to illustrate the feasibility and effectiveness of the obtained theoretical results.**

***Index Terms*—Directed spanning tree, local consensus, multi-agent systems, nonlinear dynamics, random switching, second-order consensus, time-delay (-free) coupling.**

I. Introduction

IN RECENT years, there has been an increasing interest in the study of the interplay between communication and control in networks [1]–[17]. In particular, coordination control of multi-agent systems has received compelling attentions from scientific communities, and emerged as a challenging new research field [1]–[6]. Coordination control means that local communication and cooperation among individual agents in the network may lead to certain desirable global behaviors. The local interaction mechanism also frequently appears in nature such as synchronization flashing of fireflies [18], movement of a school of fish [18], descriptions of the heart [19], the understanding of brain seizures [20], nonlinear optics [21], meteorology [22] and so on. These collective activities of creatures have inspired the designs of many practical engineering applications [6], including (but not being limited to) the formation control of multi-robots [23] and unmanned autonomous vehicles (UAV) [24] in control engineer, the distributed computation [25] and the coordination control of distributed sensor networks [26], [42], [43] in computer science, swarming or flocking [27], complex networks [8]–[13], to name a few. A fundamental approach to make the states of multi-agent systems reach an agreement on a common value of interest is consensus analysis. This is mainly because consensus analysis not only helps in better understanding the general mechanisms and interconnection rules of natural collective phenomena, but also benefits many practical applications of networked cyber-physical systems.

When the control input is added on the velocity term, each agent can be modeled simply as a first-order integrator. Significant progresses have been made towards the consensus problem of multi-agent systems for this case, see [1], [2], [4], to name a few. It is also noted that almost all the reported works can be treated as a special case of the synchronization problem of complex dynamical networks [28], [39], [40], [44], [45], [49] which has been widely studied in the past decades [8]–[13]. However, as pointed out in [29], the extension of consensus algorithms for agents from first-order dynamics to second-order (when the control input is added on the driving force/acceleration term) is non-trivial. Generally, for the following two cases that there contains a directed spanning tree for a fixed topology and all the topologies in the switching sequence contain a directed spanning tree, the existing first-order consensus criteria can be extended to handle with first-order consensus problems. While the obtained first-order consensus criteria fails to solve the consensus problem for second-order multi-agent systems when there exist isolated agents in some topologies in the switching sequence. Very recently, the second-order consensus of multi-agent systems has attracted more and more attention under various assumptions, such as communication time-delays [3], [31], [32], switching topology [5], [7], [32], [33], nonlinear dynamics [28], [30], [34], [35], [50]–[52], or

Manuscript received January 13, 2012; revised April 12, 2012; accepted October 04, 2012. Date of publication May 17, 2013; date of current version May 23, 2013. This work was supported in part by Project No. CDJXS12180002 funded by the Fundamental Research Funds for the Central Universities, in part by the Scholarship Award for Excellent Doctoral Student granted by Ministry of Education (No. 0903005109081-011), in part by the National Natural Science Foundation of China under Grant 60973114 and Grant 61170249, in part by the Research Fund of Preferential Development Domain for the Doctoral Program of Ministry of Education of China under Grant 201101911130005, in part by the State Key Laboratory of Power Transmission Equipment & System Security and New Technology, Chongqing University, under Grant 2007DA10512711206, in part by the Program for Changjiang Scholars, and in part by the National Priority Research Project NPRP 4-1162-1-181 funded by Qatar National Research Fund. This paper was recommended by Associate Editor I. Belykh.

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Digital Object Identifier 10.1109/TCSI.2012.2226507

nonlinear coupling [34], leader [28], [32], [33] or leaderless [3], [5], [7], [30], [34], [35], and communication noises [3], [32].

Usually, in reality, some oscillators, for example, harmonic oscillators [34], [36] and pendulums [37], are governed by second-order systems with both position and velocity terms. Hence, it is necessary to investigate the consensus problem of multi-agent systems composed of second-order oscillators, in which the dynamics of each agent are not only determined by the interactions among agents, but also by its own more complicated dynamics, i.e., intrinsic dynamics [28], [30], [34], [35] (disturbances and unmodeled uncertainties of agent). Protocols or algorithms dealing with the second-order consensus of multi-agent systems with nonlinear dynamics have not been emphasized until the recent works [28], [30], [34]. In [30], a kind of measurement for directed strongly connected graph, i.e., general algebraic connectivity, was first defined by Yu *et al.* The authors built the bridge between the general algebraic connectivity and the performance of reaching an agreement for second-order multi-agent systems with nonlinear dynamics. The similar problem has also been paid attention by Song *et al.* in [28] by using pinning control technique, and it is worth mentioning that the above approaches have overcome the restriction, i.e., the interaction network is strongly connected in [30]. In [34], based on the local adaptive strategies, Su *et al.*, have found that if one agent has access to the information of the virtual leader, all agents in the group can synchronize with the virtual leader. However, there exists a common drawback in the previous works: the network topology is deterministic or static, and the inner coupling matrix is constant in time. In real-world applications such as the case of terrestrial planet finder (TPF) mission and other similar mission scenarios, the sensing and inter-spacecraft communication topology often changes over time due to the dynamic nature of each spacecraft's state (e.g., range limitations on relative sensing, shadowing scenario, etc.) [1]. Hence, the study on the consensus problem of multi-agent systems with switching topology under the removal of old links and/or the addition of new links with mobile nodes, is not only important but also necessary [35]. Generally speaking, the consensus of multi-agent systems with switching topologies can be divided into the following cases: arbitrary switching [33], [38] Markov switching [5], controlled switching [32], [35] and random switching [1], [2], [13], [14], [41]. Random switching means communication among agents in the network is dependent on a time-varying topology which may vary randomly based on a pre-given probability matrix. Moreover, the other switching modes may be regarded as special cases of random switching in which some special switching sequences may take place.

In addition, problems on directed graphs are theoretically more challenging than those on undirected graphs due to the fact that algebraic properties are mostly known for undirected graphs [4]. Also, many important real networks have directed edges [46]. Moreover, with increasingly strict requirements for control speed and system performance, the unavoidable time delays in both controllers and actuators have also become a serious problem. For instance, all digital controllers, analogue anti-aliasing and reconstruction filters have also exhibited a certain time delay during operation, and the hydraulic actuators and human being interaction usually show even more significant time delays. To the best of the authors' knowledge, few authors have considered the second-order dynamic consensus problem for multi-agent systems with time-delay (-free) coupling over random switching directed networks (that communicate via a stochastic information network) thus far. Motivated by the above discussions, this challenging scenario will be investigated in this paper. Communication among agents is modeled as a randomly directed graph with different edge weights which switches with a fixed period. The existence of any edge is probabilistic and independent of the existence of any other edge. By applying the orthogonal decomposition method, the system state vector can be decomposed as two transversal components, one of which evolves along the consensus manifold and the other evolves transversally with the consensus manifold. Several sufficient conditions for almost sure consensus are derived for the cases of time-delay-free coupling and time-delay coupling, respectively. For the case of time-delay-free coupling, we find that if there exists one directed spanning tree in the network which corresponds to the fixed time-averaged topology and the switching rate of the dynamic network is not more than a critical value which is also estimated analytically, then second-order dynamic consensus can be guaranteed for the choice of suitable parameters. For the case of time-delay coupling, we not only prove that under some assumptions, the second-order consensus can be reached exponentially, but also give an analytical estimation of the upper bounds of convergence rate and the switching rate. Moreover, the obtained results are quite powerful, and can be further used to solve various switching cases for complex dynamical networks. Finally, numerical simulations are provided to illustrate the feasibility and effectiveness of the obtained theoretical results.

The rest of this paper is structured as follows: Section II introduces some basic concepts of random graph, and formulates the problem under investigation. In Section III, several sufficient conditions for reaching second-order dynamic consensus over random switching networks are derived for the cases of time-delay-free coupling and time-delay coupling, respectively. Our main results are illustrated by numerical simulations in Section IV. Concluding remarks and future research topics are drawn in Section V.

Notation: We list some mathematical notations used in this paper. Let Z^+ be the set of positive integer numbers. R^n and $R^{n\times m}$ denote the n-dimensional Euclidean space and the set of $n\times m$ real matrices, respectively. For a vector $u\in R^n$, its Euclidean norm is defined as $\|u\|=(u^{\mathrm{T}}u)^{1/2}$ and $\|u(t)\|_\tau=\sup_{-\tau\le\theta\le 0}\|u(t+\theta)\|$. I_n and O_n denote the identity matrix and zero matrix of order n, respectively. $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimum and maximum eigenvalue of a square matrix A, respectively. $\lambda_i(A)$ denotes the i-th eigenvalue of a square matrix A. Let $e=[1,1,\ldots,1]^{\mathrm{T}}$. The Kronecker product, denoted by $\otimes$, facilitates the manipulation of matrices with appropriate dimensions by the following properties: i) $(A\otimes B)(C\otimes D)=AC\otimes BD$ and ii) $(A\otimes B)^{\mathrm{T}}=A^{\mathrm{T}}\otimes B^{\mathrm{T}}$. The symmetric part of a $B\in R^{m\times m}$ is indicted with $sym(B)=1/2(B+B^{\mathrm{T}})$. $\mathrm{Re}(\lambda)$ and $\mathrm{Im}(\lambda)$ denote the real and imaginary part of a complex number λ, respectively.

II. Preliminaries and Problem Formulations

A. Random Graph

Graph theory is the study of objects, naturally called graphs, consisting of a set of vertices, each pair of which is endowed with an incidence relation represented by an edge. In variety of emerging applications, including mobile ad hoc networks, opinion dynamics, cooperative control, mathematical epidemiology, information exchange among agents in multi-agent networked systems can be modeled by an interaction random graph. Mathematically, a dynamically switching directed/undirected random network can be described by a sequence of weighted directed/undirected random graphs $G=(V,E,W_e)$, where $V=\{1,2,\ldots,N\}$ is a vertex set whose elements denote the agents in the networks; $E\subseteq V\times V$ is an edge set whose elements denote the directed/undirected communication links between agents. A directed edge $E_{ij}\in E$ in graph G is represented by an ordered pair of (i,j), where i is the head and j is the tail, which also means that vertex i can receive information from vertex j for $i,\, j\in V$. We assume that the existence of an edge from vertex i to vertex $j(\neq i)$ in graph $G(t)$ is determined randomly and is also independent of other edges with probability p_{ij} which satisfies $0\leq p_{ij}\leq 1$. An information link is referred to as a potential link when the associated edge probability $p_{ij}>0$. The probabilities p_{ij}'s are collected in the probability matrix $P=[p_{ij}]$. We also assume that the random graph does not have self-loops, i.e., no single edge starts and ends at the same vertex. Thus we have $p_{ij}=0$ for $i\in V$. We then define $N(N-1)$ independent Bernoulli random variables, δ_{ij}'s, $i,\, j\in V,\, i\neq j$, as follows: $\delta_{ij}=1$ with probability p_{ij} and $\delta_{ij}=0$ with probability $1-p_{ij}$, in which each random variable δ_{ij} is associated with the edge (i,j). $W_e=[W_{eij}]$ is a weight matrix with all diagonal elements equal to 0, and the element W_{eij} denotes the weight associated with the edge (i,j). The weight denotes how each agent evaluates the information collected from its neighboring agents to update the consensus algorithm. We have that W_e is symmetric for undirected graphs while it can be asymmetric for directed graphs. Moreover, a directed graph has a directed spanning tree if there exists at least one vertex called root which has a directed path to all the other vertices.

Algebraically, a weighted directed random graph $G(t)$ is represented by an adjacency matrix $A=[a_{ij}]$ and a Laplacian matrix $L=[L_{ij}]$ defined as $a_{ij}=0$ if $i=j$ and $a_{ij}=W_{eij}\delta_{ij}$ if $i\neq j$, where W_{eij} is the corresponding entry of the weight matrix W_e, and $L_{ij}=\sum_{k=1}^{N}a_{ik}$ if $i=j$ and $L_{ij}=-a_{ij}$ if $i\neq j$ [4], [13]. Both the adjacency matrix and the Laplacian matrix defined above are essentially random. The Laplacian matrix L is a zero row-sum matrix, therefore e is an eigenvector of L associated with the eigenvalue 0. In addition, the rank of L equals to $N-1$ if and only if, for an undirected graph G, G is connected; for a directed graph G, G has a directed spanning tree. In one of the circumstances, the spectrum of L can be ordered of the form: $0=\mathrm{Re}\lambda_1(L)<\mathrm{Re}\lambda_2(L)\leq\cdots\leq\mathrm{Re}\lambda_{N-1}(L)\leq\mathrm{Re}\lambda_N(L)$ [1]. As shown in [13], the authors also consider a class of random graph $G(t)$, which keeps unchanged in the interval $t\in[(k-1)\Delta,k\Delta)$ and switches at a series of fixed time instants $\{k\Delta,\, k\in Z^+\}$ where Δ is called fixed period or switching rate. The finite sample space of the random directed graph is indicted by $\mathcal{G}$, and the elementary events (possible graphs) are indicted by $G^{(j)}$, $j=1,2,\ldots,|\mathcal{G}|$ where $|\mathcal{G}|$ represents cardinality. The Laplacian matrix which corresponds graph $G^{(j)}$ is denoted as $L^{(j)}$. In this sense, a multi-agent system which corresponds to a random switching network can be viewed as a set of nonlinear stochastically switched systems. It follows from the switching mechanisms described above that the graph edges are independent random variables. The fixed time-averaged topology of the random graph Laplacian matrix, written $E[L]=[E[L_{ij}]]$, may be computed entrywise $E[L_{ij}]=-p_{ij}W_{eij}$ for $i\neq j$ and $E[L_{ij}]=\sum_{k=1}^{N}p_{ik}W_{eik}$ for $i=j$. And the fixed time-averaged topology, i.e., $E[L]$, corresponds to a weighted directed graph which does not necessarily belong to $\mathcal{G}$. We refer to this graph as the average graph, denoted by $E[G]$ as in [13].

B. Problem Formulations

In this paper, we consider a dynamical network $G(t)$ composed of N identical agents with second-order nonlinear dynamics. Suppose $G(t)$ is interconnected pairwise via a random, weighted, directional time-delay states information interaction, in which each agent is an n-dimensional dynamical unit. The model of each agent can be described by

$$\begin{cases}\dot{x}_i(t)=v_i(t),\\ \dot{v}_i(t)=f(x_i(t),v_i(t),t)-\alpha\sum\limits_{j=1}^{N}L_{ij}(\frac{t}{\Delta})B(t)x_j(t-\tau)\\ \qquad -\beta\sum\limits_{j=1}^{N}L_{ij}(\frac{t}{\Delta})B(t)v_j(t-\tau), i=1,2,\ldots,N,\end{cases}\tag{1}$$

where $x_i(t)=(x_{i1}(t),\ldots,x_{in}(t))^{\mathrm{T}}\in R^n$ and $v_i(t)=(v_{i1}(t),\ldots,v_{in}(t))^{\mathrm{T}}\in R^{\mathrm{n}}$ are the position and velocity vector of the i-th agent, respectively; $f:R^n\times R^n\times R^+\rightarrow R^n$ is a continuously differentiable vector-valued function, which describes the agents' individual dynamics;$\alpha>0$ and $\beta>0$ stand for position and velocity coupling strengths between any two agents in the network, that partially assigns coupling strength between agents; $B(t)\in R^{n\times n}$ is a semi-positive definite matrix, modeling the time-varying inner coupling among agents; $L(t/\Delta)=(L_{ij}(t/\Delta))_{N\times N}$ is the Laplacian matrix representing the topological structure of the random network $G(t)$ at time t. $\tau\geq 0$ specifies the coupling delay between agents. Here, t and t/Δ denote two kinds of distinct time scales.

For system (1), if $x_i(t)=s_1(t)$ and $v_i(t)=s_2(t)$, $i=1,2,\ldots,N$, and some $S(t)=[s_1^{\mathrm{T}}(t),s_2^{\mathrm{T}}(t)]\in R^{2n}$ is a solution of the individual subsystem

$$\begin{cases}\dot{s}_1(t)=s_2(t),\\ \dot{s}_2(t)=f(s_1(t),s_2(t),t),\end{cases}\tag{2}$$

then we can see that the second-order dynamical consensus can be achieved. Here, $S(t)$ is called consensus manifold. Generally, $S(t)$ can be an equilibrium point, a nontrivial periodic orbit, or even a chaotic attractor defined for finite dimensional systems. It is also noted that the consensus we discuss in this paper refers to the almost sure consensus over random switching directed network $G(t)$. Almost sure consensus is also called consensus with probability one [4].

The problem of second-order consensus for multi-agent systems with nonlinear dynamics that communicate via a stochastic information network corresponds to stability analysis of consensus manifold in the randomly coupled dynamical network (1). To this end, subtracting (2) from (1) and noticing that the all of the row sums of $L(t/\Delta)$ equal to one, we obtain the following error dynamical system

$$\begin{cases} \dot{\hat{x}}_i(t) = \hat{v}_i(t), \\ \dot{\hat{v}}_i(t) = f(x_i(t), v_i(t), t) - f(s_1(t), s_2(t), t) \\ \qquad -\alpha \sum_{j=1}^{N} L_{ij}(\frac{t}{\Delta})B(t)\hat{x}_j(t-\tau) \\ \qquad -\beta \sum_{j=1}^{N} L_{ij}(\frac{t}{\Delta})B(t)\hat{v}_j(t-\tau), \end{cases} \tag{3}$$

where $\hat{x}_i(t) = x_i(t) - s_1(t)$ and $\hat{v}_i(t) = v_i(t) - s_2(t)$, $i = 1, 2, \ldots, N$. Linearzing (3) around the consensus manifold $S(t)$ leads to

$$\begin{cases} \dot{\hat{x}}_i(t) = \hat{v}_i(t), \\ \dot{\hat{v}}_i(t) = D_x f(s_1, s_2, t)\hat{x}_i(t) + D_y f(s_1, s_2, t)\hat{v}_i(t) \\ \qquad -\alpha \sum_{j=1}^{N} L_{ij}(\frac{t}{\Delta})B(t)\hat{x}_j(t-\tau) \\ \qquad -\beta \sum_{j=1}^{N} L_{ij}(\frac{t}{\Delta})B(t)\hat{v}_j(t-\tau) \\ \qquad +O_i(\hat{x}_i(t), t) + O_i(\hat{v}_i(t), t), i = 1, 2, \ldots, N, \end{cases} \tag{4}$$

where $D_x f(s_1, s_2, t)$ and $D_y f(s_1, s_2 t)$ denote the Jacobian matrices of function $f(x, y, t)$ towards the state vectors x and y on the consensus manifold $S(t)$, respectively. In addition, the high-order terms satisfy $\lim_{\|\hat{x}_i(t)\| \to 0}(\|O_i(\hat{x}_i(t), t)\|/\|\hat{x}_i(t)\|) = 0$ and $\lim_{\|\hat{v}_i(t)\| \to 0}(\|O_i(\hat{v}_i(t), t)\|/\|\hat{v}_i(t)\|) = 0$.

Equivalently, (4) can be rewritten as the following compact vector form:

$$\dot{y}(t) = (I_N \otimes F(s_1, s_2, t))\, y(t) - \left(\tilde{L}\left(\frac{t}{\Delta}\right) \otimes B(t)\right) \times y(t-\tau) + O(y(t), t), \tag{5}$$

where $y(t) = (\hat{x}^{\mathrm{T}}(t), \hat{v}^{\mathrm{T}}(t))^{\mathrm{T}}$, $\hat{x}(t) = [\hat{x}_1^{\mathrm{T}}(t), \hat{x}_2^{\mathrm{T}}(t), \ldots, \hat{x}_N^{\mathrm{T}}(t)]^{\mathrm{T}}$, $\hat{v}(t) = [\hat{v}_1^{\mathrm{T}}(t), \hat{v}_2^{\mathrm{T}}(t), \ldots, \hat{v}_N^{\mathrm{T}}(t)]^{\mathrm{T}}$, $F(s_1, s_2, t) = \begin{bmatrix} O_n & I_n \\ D_x f(s_1, s_2, t) & D_y f(s_1, s_2, t) \end{bmatrix}$ and $\begin{bmatrix} O_N & O_N \\ \alpha L(t/\Delta) & \beta L(t/\Delta) \end{bmatrix}$. Similarly, the high-order term also satisfies $\lim_{\|y\| \to 0}(\|O(y, t)\|/\|y\|) = 0$. It follows that analyzing the asymptotical stability of linear part of random switching system (5) at the origin suffices to investigate the second-order dynamic local consensus of all agents over the random switching directed network $G(t)$.

III. Main Results

A. Orthogonal Decomposition

In this section, we will give some results regarding to the time-delay-free case ($\tau = 0$) and the time-delay case ($\tau > 0$), respectively. In general, to investigate the second-order dynamic local consensus of all agents in the random switching network $G(t)$, we need to consider the local stability of error system (5) along the consensus manifold $S(t)$. To begin with, we first decompose the state vector of (5) (neglecting the high-order term $O(y(t), t)$ into two components which are orthogonal each other [9], [13], [14]. One component evolves along the consensus manifold, and the other evolves transverse to the consensus manifold. Since $e \in R^N$, we denote its spanned subspace by $\mathbf{A}$. On the other hand, each subspace of R^N has only one orthogonal complementary subspace, so the orthogonal subspace of $\mathbf{A}$ uniquely exists. Suppose that this orthogonal complementary space $\mathbf{A}^{\perp}$ is the column space of matrix $W \in R^{N\times(N-1)}$ that satisfies $W^{\mathrm{T}}e = 0$ and $W^{\mathrm{T}}W = I_{N-1}$, where $W = (W_2, W_3, \ldots, W_N)$ consists of an array of $N-1$ vectors in R^N. (In the following section, we will discuss the existence of orthogonal decomposition matrix W). We therefore have $R^N = \mathbf{A} \oplus \mathbf{A}^{\perp}$. Similarly, we can also expand this decomposition into R^{nN}, i.e., $R^{nN} = \mathbf{B} \oplus \mathbf{B}^{\perp}$, where $\mathbf{B}$ is the subspace spanned by $e \otimes I_n$ and $\mathbf{B}^{\perp}$ represents the orthogonal complement space spanned by $W \otimes I_n$. Note that the consensus state $e \otimes s(t) = 0$ is in the range of $e \times I_n = 0$ and then in the null space of $W^{\mathrm{T}} \otimes I_n$ [9]. So the state variable $y(t) \in R^{2nN}$ can be decomposed into a component in the subspace $\mathbf{B}$ spanned by $e \otimes I_n = 0$ and a component in the subspace $\mathbf{B}^{\perp}$ spanned by $W^{\mathrm{T}} \otimes I_n$ as follows:

$$y(t) = e \otimes \bar{y}(t) + (W \otimes I_{2n})\eta(t), \tag{6}$$

where $\bar{y}(t) = 1/N(e \otimes I_{2n})^{\mathrm{T}} y(t) \in R^{2n}$ and $\eta(t) = (W \otimes I_{2n})^{\mathrm{T}} \times y(t) \in R^{2n(N-1)}$. Note that $\bar{y}(t)$ is the average of all the components in $y(t)$, and the two components are orthogonal each other, i.e.,

$$\begin{aligned} &[(W \otimes I_{2n})\eta(t)]^{\mathrm{T}} (e \otimes \bar{y}(t)) \\ &\quad = \eta^{\mathrm{T}}(t)(W^{\mathrm{T}} \otimes I_{2n}) (e \otimes \bar{y}(t)) \\ &\quad = \eta^{\mathrm{T}}(t) \left(W^{\mathrm{T}} e \otimes \bar{y}(t)\right) \\ &\quad = 0. \end{aligned} \tag{7}$$

Using the following state transformation

$$\begin{pmatrix} \bar{y}(t) \\ \eta(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{N}(e \otimes I_{2n})^{\mathrm{T}} \\ (W \otimes I_{2n})^{\mathrm{T}} \end{pmatrix} y(t), \tag{8}$$

the linear part of (5) can be partitioned as two dynamically coupled subsystems. The first subsystem can be described as:

$$\begin{aligned} \dot{\bar{y}}(t) &= \frac{1}{N}(e \otimes I_{2n})^{\mathrm{T}} \left[(I_N \otimes F(s_1, s_2, t))\, y(t) - \left(\tilde{L}\left(\frac{t}{\Delta}\right) \otimes B(t)\right) y(t-\tau) \right] \\ &= \frac{1}{N}(e^{\mathrm{T}} \otimes F(s_1, s_2, t)) y(t) - \frac{1}{N}(e^{\mathrm{T}} \otimes I_{2n}) \left(\tilde{L}\left(\frac{t}{\Delta}\right) \otimes B(t)\right) y(t-\tau) \\ &= \frac{1}{N}(I_N \otimes F(s_1, s_2, t))(e^{\mathrm{T}} \otimes I_{2n}) y(t) - \frac{1}{N}(e^{\mathrm{T}} \otimes I_{2n}) \left(\tilde{L}\left(\frac{t}{\Delta}\right) \otimes B(t)\right) (e \otimes \bar{y}(t-\tau) \\ &\quad + (W \otimes I_{2n})\eta(t-\tau)) \\ &= (I_N \otimes F(s_1, s_2, t))\bar{y}(t) - \Delta_1 - \Delta_2, \end{aligned} \tag{9}$$

where

$$\Delta_1 = \frac{1}{N}(e^{\mathrm{T}} \otimes I_{2n}) \left(\tilde{L}\left(\frac{t}{\Delta}\right) \otimes B(t)\right) (e \otimes \bar{y}(t-\tau))$$

$$
\begin{aligned}
&= \frac{1}{N^2}(e^{\mathrm{T}} \otimes I_{2n}) \left[\begin{pmatrix} O_N & O_N \\ \alpha L\left(\frac{t}{\Delta}\right) & \beta L\left(\frac{t}{\Delta}\right) \end{pmatrix} \otimes B(t) \right] \\
&\quad \times \left[e \otimes \left((e^{\mathrm{T}} \otimes I_{2n}) y(t-\tau) \right) \right] \\
&= \frac{1}{N^2}(e^{\mathrm{T}} \otimes I_{2n}) \left[\begin{pmatrix} O_N & O_N \\ \alpha L\left(\frac{t}{\Delta}\right) & \beta L\left(\frac{t}{\Delta}\right) \end{pmatrix} \otimes B(t) \right] \\
&\quad \times \begin{bmatrix} e \otimes \left((e^{\mathrm{T}} \otimes I_n) \hat{x}(t-\tau) \right) \\ e \otimes \left((e^{\mathrm{T}} \otimes I_n) \hat{v}(t-\tau) \right) \end{bmatrix} \\
&= \frac{1}{N^2}(e^{\mathrm{T}} \otimes I_{2n}) \left[\begin{matrix} 0_{nN} \\ \alpha L\left(\frac{t}{\Delta}\right) e \otimes B(t) \left((e^{\mathrm{T}} \otimes I_n) \hat{x}(t-\tau) \right) \\ + \qquad 0_{nN} \\ + \beta L\left(\frac{t}{\Delta}\right) e \otimes B(t) \left((e^{\mathrm{T}} \otimes I_n) \hat{v}(t-\tau) \right) \end{matrix} \right] \\
&= \frac{1}{N^2}(e^{\mathrm{T}} \otimes I_{2n}) \begin{pmatrix} 0_{nN} \\ 0_{nN} \end{pmatrix} \\
&= 0_{2n}
\end{aligned} \tag{10}
$$

and

$$
\Delta_2 = \frac{1}{N}(e^{\mathrm{T}} \otimes I_{2n}) \left(\tilde{L}\left(\frac{t}{\Delta}\right) \otimes B(t) \right) (W \otimes I_{2n}) \eta(t-\tau). \tag{11}
$$

Therefore,

$$
\begin{aligned}
\dot{\bar{y}}(t) = (I_N \otimes F(s_1, s_2, t)) \bar{y}(t) - \frac{1}{N}(e^{\mathrm{T}} \otimes I_{2n}) \\
\times \left(\tilde{L}\left(\frac{t}{\Delta}\right) \otimes B(t) \right) (W \otimes I_{2n}) \eta(t-\tau).
\end{aligned} \tag{12}
$$

Similarly, the second subsystem can be expressed as:

$$
\begin{aligned}
\dot{\eta}(t) &= (W \otimes I_{2n})^{\mathrm{T}} \dot{y}(t) \\
&= (W^{\mathrm{T}} \otimes I_{2n}) \Bigg[(I_N \otimes F(s_1, s_2, t)) \, y(t) \\
&\qquad - \left(\tilde{L}\left(\frac{t}{\Delta}\right) \otimes B(t) \right) y(t-\tau) \Bigg] \\
&= \left(W^{\mathrm{T}} \otimes F(s_1, s_2, t) \right) y(t) \\
&\quad - (W^{\mathrm{T}} \otimes I_{2n})(\tilde{L}\left(\frac{t}{\Delta}\right) \otimes B(t)) y(t-\tau) \\
&= \left(W^{\mathrm{T}} \otimes F(s_1, s_2, t) \right) \\
&\quad \times \left(e \otimes \bar{y}(t) + (W \otimes I_{2n}) \eta(t) \right) - \Delta_3 \\
&= \left(W^{\mathrm{T}} e \otimes F(s_1, s_2, t) \right) \bar{y}(t) \\
&\quad + (W^{\mathrm{T}} W \otimes F(s_1, s_2, t)) \eta(t) - \Delta_3 \\
&= (I_{N-1} \otimes F(s_1, s_2, t)) \eta(t) - \Delta_3
\end{aligned} \tag{13}
$$

where

$$
\begin{aligned}
\Delta_3 &= (W^{\mathrm{T}} \otimes I_{2n})(\tilde{L}\left(\frac{t}{\Delta}\right) \otimes B(t)) y(t-\tau) \\
&= (W^{\mathrm{T}} \otimes I_{2n})(\tilde{L}\left(\frac{t}{\Delta}\right) \otimes B(t)) \\
&\quad \times [e \otimes \bar{y}(t-\tau) + (W \otimes I_{2n}) \eta(t-\tau)] \\
&= (W^{\mathrm{T}} \otimes I_{2n})(\tilde{L}\left(\frac{t}{\Delta}\right) \otimes B(t))(e \otimes \bar{y}(t-\tau)) \\
&\quad + (W^{\mathrm{T}} \otimes I_{2n})(\tilde{L}\left(\frac{t}{\Delta}\right) \otimes B(t))(W \otimes I_{2n}) \eta(t-\tau) \\
&= (W^{\mathrm{T}} \otimes I_{2n}) \begin{pmatrix} 0_{nN} \\ 0_{nN} \end{pmatrix} \\
&\quad + (W^{\mathrm{T}} \otimes I_{2n})(\tilde{L}\left(\frac{t}{\Delta}\right) \otimes B(t))(W \otimes I_{2n}) \eta(t-\tau) \\
&= (W^{\mathrm{T}} \otimes I_{2n})(\tilde{L}\left(\frac{t}{\Delta}\right) \otimes B(t))(W \otimes I_{2n}) \eta(t-\tau).
\end{aligned} \tag{14}
$$

Combining (12) and (13) yields the following two coupled random switching subsystems:

$$
\begin{cases}
\dot{\bar{y}}(t) = (I_N \otimes F(s_1, s_2, t)) \bar{y}(t) - \frac{1}{N}(e^{\mathrm{T}} \otimes I_{2n}) \\
\qquad \times (\tilde{L}\left(\frac{t}{\Delta}\right) \otimes B(t))(W \otimes I_{2n}) \eta(t-\tau), \\
\dot{\eta}(t) = (I_{N-1} \otimes F(s_1, s_2, t)) \eta(t) - (W^{\mathrm{T}} \otimes I_{2n}) \\
\qquad \times (\tilde{L}\left(\frac{t}{\Delta}\right) \otimes B(t))(W \otimes I_{2n}) \eta(t-\tau).
\end{cases} \tag{15}
$$

Correspondingly, we write the deterministic expectation system associated with the random switching system (15) as follows:

$$
\begin{cases}
\dot{\bar{y}}_1(t) = (I_N \otimes F(s_1, s_2, t)) \bar{y}_1(t) - \frac{1}{N}(e^{\mathrm{T}} \otimes I_{2n}) \\
\qquad \times (E[\tilde{L}] \otimes B(t))(W \otimes I_{2n}) \xi(t-\tau), \\
\dot{\xi}(t) = (I_{N-1} \otimes F(s_1, s_2, t)) \xi(t) - (W^{\mathrm{T}} \otimes I_{2n}) \\
\qquad \times (E[\tilde{L}] \otimes B(t))(W \otimes I_{2n}) \xi(t-\tau),
\end{cases} \tag{16}
$$

in which $E[\tilde{L}]$ denotes the expectation matrix of random switching matrix $\tilde{L}(t/\Delta)$. In the following, we will respectively discuss two cases in detail, i.e., the case of time-delay-free coupling $\tau = 0$ and the case of time-delay coupling $\tau > 0$.

B. The Case of Time-Delay-Free Coupling

First, we will give a lemma for deriving our main results regarding to the time-delay-free case. It establishes the relationship between asymptotical stability of the second differential equation in (15) and that of a derived sampled-data system (at the switching instants Δk for all $k \in Z^+$).

Lemma 1: For random switching system (15)with $\tau = 0$, suppose that $F(s_1, s_2, t))$, $\tilde{L}(t/\Delta)$ and $B(t)$ defined in (5) are bounded and piecewise continuous functions for all $t \geq 0$, and $\tilde{L}(t/\Delta)$ is constant for all $t \in [(k-1)\Delta, k\Delta)$ and switches at fixed time instants $t = \Delta k$ for all $k \in Z^+$. If the sampled sequence $\{\eta(\Delta k)\}$ convergences to zero almost surely, then for all $t \in [(k-1)\Delta,\ k\Delta)$, the state vector $\eta(t)$ will decay to zero almost surely too.

Proof: Let η_k be $\eta(k\Delta)$ for simplicity. For any $t \in [(k-1)\Delta,\ k\Delta)$, $k \in Z^+$, $\eta(t)$ can be computed by $\eta(t) = \phi_\eta(t,\ (k-1)\Delta)\eta_{k-1}$, where $\phi_\eta(t,\ \tau)$ denotes the transition matrix of $\dot{\eta}(t) = M(t)\eta(t)$ from τ to t with $M(t) = I_{N-1} \otimes F(s_1, s_2, t) - (W^{\mathrm{T}} \otimes I_{2n})(\tilde{L}(t/\Delta) \otimes B(t))(W \otimes I_{2n})$. Since $F(s_1, s_2, t)$, $\tilde{L}(t/\Delta)$ and $B(t)$ are bounded and piecewise continuous functions, thus there exist positive constants m, λ and $\bar{\beta}$ such that for any $t \geq 0$, $\|F(s_1, s_2, t))\| \leq m, \|\tilde{L}(t/\Delta)\| \leq \lambda$ and $\|B(t)\| \leq \bar{\beta}$. In addition, from the definition of matrix W, we obtain $\|W\| = \|W^{\mathrm{T}}\| = 1$. According to the Gronwall

Bellman's inequality [see 13, the proof of Theorem 1], for any $t \in [(k-1)\Delta,\ k\Delta),\ k \in Z^+, \eta(t)$ could be estimated by

$$\begin{aligned}\|\eta(t)\| &\leq \|\eta_{k-1}\| \exp\left(\int_{(k-1)\Delta}^{k\Delta} \left(m + \lambda\bar{\beta}\right) d\tau\right)\\ &= \|\eta_{k-1}\| \exp\left((m + \lambda\bar{\beta})\Delta\right), \end{aligned} \tag{17}$$

and the claim follows immediately.

Consequently, the second equation of system (16) may be compactly rewritten as:

$$\dot{\xi}(t) = \bar{M}(t)\xi(t), \tag{18}$$

where $\bar{M}(t) = I_{N-1} \otimes F(s_1, s_2, t) - (W^{\mathrm{T}} \otimes I_{2n})(E[\tilde{L}] \otimes B(t))(W \otimes I_{2n})$.

We list the following assumptions for our main results.

(H1) There exists a directed spanning tree in the fixed time-averaged graph $\bar{G}$ associated with the expectation Laplacian matrix $E\left[L\left(t/\Delta\right)\right]$.

(H2) Let $P(t)$ be a symmetric positive definite and bounded matrix, i.e., for some constants $p_2 > p_1 > 0$ such that $p_1 I \leq P(t) \leq p_2 I$ and $\bar{M}^{\mathrm{T}}(t)P(t) + \dot{P}(t) + P(t)\bar{M}(t) \leq -\mu I < 0$ for some $\mu > 0$.

(H3) $F(s_1,\ s_2,\ t))$, $\tilde{L}(t/\Delta)$ and $B(t)$ defined in (5) are bounded and piecewise continuous functions for all $t \geq 0$, i.e., there exist three positive real numbers m, λ and $\bar{\beta}$ such that $\|F(s_1,\ s_2,\ t))\| \leq m$, $\|\tilde{L}(t/\Delta)\| \leq \lambda$ and $\|B(t)\| \leq \bar{\beta}$ for all $t \geq 0$; $\tilde{L}(t/\Delta)$ is constant for all $t \in [(k-1)\Delta,\ k\Delta)$ and switches at fixed time instants $t = \Delta k$ for all $k \in Z^+$.

Based on the previous lemmas, the main results of the time-delay-free case can be stated as follows.

Theorem 1: Suppose that (H1)–(H3) hold, and $\Delta < 1/2\gamma \ln\left(\mu - 4\gamma p_2 + a/a\right)$ (γ and α are estimated in the following proof procedure). Then state vector $\eta(t)$ of stochastically dynamical system (15)with $\tau = 0$ will converge to zero almost surely, which implies that the second-order dynamical local consensus over the random switching network (1) can be achieved almost surely.

Proof: (H1) can guarantee the existence of the orthogonal decomposition matrix W, i.e., the existence of (15). Construct the quadratic function as:

$$V(\xi(t)) = \xi^{\mathrm{T}}(t)P(t)\xi(t). \tag{19}$$

The time derivative of V along the trajectory of the expectation system (18) is given by

$$\dot{V}(\xi(t)) = \xi^{\mathrm{T}}(t)\left[\bar{M}^{\mathrm{T}}(t)P(t) + \dot{P}(t) + P(t)\bar{M}(t)\right]\xi(t). \tag{20}$$

We note from (H2) and (20)that V is a Lyapunov function for the expectation system (18) whose time derivative along the flow of (18) is strictly negative definite. Thus, the expectation system (18) is globally exponentially stable. However, in general, the function $V(\eta(t))$ is not a Lyapunov function for the stochastically switched system (20). Inspired by the results of Porfiri *et al.* in [13], we use the difference method to complete this task. For every $k \in Z^+$, we define

$$\begin{aligned}\Delta V(\eta(t), (k+1)\Delta,\ k\Delta) &= V(\eta_{k+1}) - V(\eta_k)\\ &= \int_{k\Delta}^{(k+1)\Delta} \dot{V}(\eta(t))dt. \end{aligned} \tag{21}$$

Let $\phi_\eta(t, t_0)$ be the transition matrix of the stochastically switched system $\dot{\eta}(t) = M(t)\eta(t)$ over the time interval $[t_0, t)$. Thus, we have $\eta(t) = \phi_\eta(t, t_0)\eta(t_0)$. Recalling the Peano-Baker expansion for $\phi_\eta(t, t_0)$ [13], [14], we have

$$\begin{aligned}\phi_\eta(t, t_0) &= I_{2(N-1)n} + \phi'_\eta(t, t_0)\\ &= I_{2(N-1)n} + \int_{t_0}^{t} M(\sigma_1)d\sigma_1\\ &\quad + \int_{t_0}^{t} M(\sigma_1)\int_{t_0}^{\sigma_1} M(\sigma_2)d\sigma_2 d\sigma_1\\ &\quad + \sum_{i=3}^{\infty}\int_{t_0}^{t} M(\sigma_1)\int_{t_0}^{\sigma_1}\\ &\quad \cdots \int_{t_0}^{\sigma_{i-1}} M(\sigma_i)d\sigma_i \ldots d\sigma_1, \end{aligned} \tag{22}$$

where $\phi'_\eta(t, t_0)$ satisfies

$$\|\phi'_\eta(t, t_0)\| \leq \gamma(t - t_0) + \frac{\gamma^2}{2!}(t - t_0)^2 + \frac{\gamma^3}{3!}(t - t_0)^3 + \cdots +, \tag{23}$$

in which $\gamma = \max\left\{\sup_{t \geq 0}\left(\|M(t)\|, \|\bar{M}(t)\|\right)\right\}$.

If $t \in [k\Delta,\ (k+1)\Delta),\ k \in Z^+$ we get $\|\phi'_\eta(t, k\Delta)\| \leq \exp(\gamma\Delta) - 1$.

Therefore, we have

$$\begin{aligned}&\Delta V(\eta(t), (k+1)\Delta, k\Delta)\\ &= \eta_k^{\mathrm{T}}\left(\int_{k\Delta}^{(k+1)\Delta}\left[M^{\mathrm{T}}(t)P(t) + \dot{P}(t) + P(t)M(t)\right]dt\right)\eta_k\\ &\quad + \eta_k^{\mathrm{T}}\tilde{Q}\eta_k, \end{aligned} \tag{24}$$

where

$$\begin{aligned}\tilde{Q} =& \\ &\int_{k\Delta}^{(k+1)\Delta}\left(\phi'_\eta(t, k\Delta)\right)^{\mathrm{T}}\left[M^{\mathrm{T}}(t)P(t) + \dot{P}(t) + P(t)M(t)\right]dt\\ &+ \int_{k\Delta}^{(k+1)\Delta}\left[M^{\mathrm{T}}(t)P(t) + \dot{P}(t) + P(t)M(t)\right]\phi'_\eta(t,\ k\Delta)dt\\ &+ \int_{k\Delta}^{(k+1)\Delta}\left(\phi'_\eta(t, k\Delta)\right)^{\mathrm{T}}\left[M^{\mathrm{T}}(t)P(t) + \dot{P}(t) + P(t)M(t)\right]\\ &\times \phi'_\eta(t, k\Delta)dt. \end{aligned} \tag{25}$$

Let $a = \sup_{t \geq 0}\left(\|M^{\mathrm{T}}(t)P(t) + \dot{P}(t) + P(t)M(t)\|\right)$, by some calculations, we can get

$$\|\tilde{Q}\| \leq 2\left(\exp(\gamma\Delta) - 1\right)\Delta a + \left(\exp(\gamma\Delta) - 1\right)^2\Delta a. \tag{26}$$

Moreover, see (27) at the bottom of the page. Therefore, the following inequality holds.

$$\Delta V(\eta(t),(k+1)\Delta,k\Delta) \\ \leq -\left[\mu-4\gamma p_2-a\left(\exp(2\gamma\Delta)-1\right)\right]\Delta\|\eta_k\|^2. \quad (28)$$

Thus, if the switching rate satisfies $\Delta < 1/2\gamma\ln\left(\mu-4\gamma p_2+a/a\right)$ the stochastic sequence $\{\eta(k\Delta),k\in Z^+\}$ will almost surely converge to zero. It follows from (H2) that the conditions in Lemma 1 are satisfied. By Lemma 1, we have that $\eta(t)$ will also decay to zero almost surely when $t\to\infty$. Therefore, the second-order dynamical local consensus over the random switching network (1) can be achieved almost surely. The proof is thus completed.

Remark 1: The results of our paper are based on the orthogonal decomposition method. Therefore, the existence of the decomposition matrix $W\in R^{N\times(N-1)}$ satisfying $W^{\mathrm{T}}e=0$ and $W^{\mathrm{T}}W=I_{N-1}$ is very essential. (H1) is indispensable because it can guarantee the existence of the decomposition matrix. W is also indispensable in testing the condition (H2). In short, (H1) can keep the existence of (15), while (H2) can guarantee the asymptotical stability of (15). According to (H1) we can seek a matrix W that satisfies the desired requirements. First, we give a fact that for a directed graph G with a directed spanning tree the eigenvalues corresponding to its Laplacian matrix L can be ordered as $0=\mathrm{Re}\lambda_1(L)<\mathrm{Re}\lambda_2(L)\leq\cdots\leq\mathrm{Re}\lambda_N(L)$. e is an eigenvector of L associated with the eigenvalue 0. Due to the asymmetry of L the eigenvectors of L associated with the $N-1$ nonzero eigenvalues are complex. They could not be used to construct W because W is needed to be a real matrix. Second, a directed graph with a directed spanning tree corresponds to a connected undirected graph by doubling the orientation of all the directed edges and taking half of weights of a directed edge as the weights of the two undirected edges. The resultant undirected graph is connected. For the obtained undirected connected graph, its weighted matrix is symmetric. The $N-1$ eigenvectors associated with $N-1$ nonzero eigenvalues of the obtained connected graph's Laplacian matrix L are real vectors. Then we can use the $N-1$ normalized real eigenvectors to construct the decomposition matrix W.

Remark 2: The idea in the above proof originated from [13]. Here we show that by suitable adapting the methods of proof in [13], and adding further substantial arguments, now allows us to rigorously estimate the upper bound of the switching rate Δ. Nevertheless, in [13] the authors have only proven that there exists a Δ^* such that for all $\Delta\leq\Delta^*$, the synchronization of complex dynamical random switching networks can be achieved.

Remark 3: From (H2) we can find a positive definite matrix $P(t)$ whose derivative matrix $\dot{P}(t)$ is negative definite to guarantee the condition is satisfied. Additionally, choose $P(t)$ to be diagonal can simply our computation. The diagonal elements $P_{ii}(t)$ may be in the form of $b_1+c_1\exp(-d_1t)$, where b_1, c_1 and d_1 are three positive constants to be determined by the inequality. At this time, we can select $p_1=b_1$ and $p_2=b_1+c_1$.

C. The Case of Time-Delay Coupling

Note that for linear systems, the uniformly asymptotical stability of solutions is equivalent to the uniformly exponential stability [9], [14]. Thus, assume that the second differential equation in (15) is uniformly asymptotically stable almost surely, by the definition of the exponential stability, for $\forall\,\varepsilon>0$, there always exists two constants $\rho>0$ and $\varphi(\varepsilon)>0$ such that for $\forall\ t>t_0\geq 0$ and $|\eta(t_0)|<\varphi$, $\eta(t)<\varepsilon\exp(-\rho t)\|\eta(t_0)\|$ holds. This also implies $\eta(t-\tau)<\varepsilon\exp(-\rho(t-\tau))\|\eta(t_0)\|$. Therefore, $\eta(t,\ t_0)$ and $\eta(t-\tau,\ t_0)$ will simultaneously almost surely converge to zero as $t\to\infty$. Then the transversal component will disappear and the second-order dynamical local consensus over the random switching directed network (1) with time-delay couplings can be achieved almost surely.

We list the following assumption for obtaining the main results of the time-delay coupling case.

(H4) There exists a infinite subsequence composed of some fixed moments in the random switching sequence, denoted $t_1^*<t_2^*<\cdots<t_{k-1}^*<t_k^*<\cdots$, and satisfies $\lim_{k\to+\infty}t_k^*=+\infty$. Moreover, there is a finite positive integer κ such that $t_{i+1}^*-t_i^*\leq\kappa\Delta$ and $V_\tau(\eta(t_{k_0+1}^*),\ t_{k_0+1}^*)-V_\tau(\eta(t_{k_0}^*),\ t_{k_0}^*)\leq -p_3\left\|\eta(t_{k_0}^*)\right\|_\tau^2<0$ where $V(\eta(t_{k_0}^*),\ t_{k_0}^*)$ is defined in the following (30), $V_\tau(\eta(t_{k_0}^*),\ t_{k_0}^*)$ is short for $\left\|V(\eta(t_{k_0}^*),\ t_{k_0}^*)\right\|_\tau$, $i=2,\ 3,\ \ldots$, and $p_3>0$.

The main results of the time-delay case can be stated as follows.

Theorem 2: Suppose that (H1)–(H4) hold. If $\tau<\tau^*=\mu p_1/2p_2^2\bar{\beta}\lambda\left(m+\bar{\beta}\lambda\right)$ and $\Delta\leq\Delta^*=\ln\left(p_2/p_3-p_2\right)/\varsigma\kappa$ with $\varsigma>0$, then state vector $\eta(t)$ of stochastically dynamical system (15) will converge to zero almost surely, which im-

$$\begin{aligned}
&\eta_k^{\mathrm{T}}\left(\int_{k\Delta}^{(k+1)\Delta}\left[M^{\mathrm{T}}(t)P(t)+\dot{P}(t)+P(t)M(t)\right]dt\right)\eta_k\\
&=\eta_k^{\mathrm{T}}\left(\int_{k\Delta}^{(k+1)\Delta}\left[\bar{M}^{\mathrm{T}}(t)P(t)+\dot{P}(t)+P(t)\bar{M}(t)\right]dt\right)\eta_k\\
&\quad+\eta_k^{\mathrm{T}}\left(\int_{k\Delta}^{(k+1)\Delta}\left[\left(M^{\mathrm{T}}(t)-\bar{M}^{\mathrm{T}}(t)\right)P(t)+P(t)\left(M(t)-\bar{M}(t)\right)\right]dt\right)\eta_k\\
&\leq-\mu\Delta\|\eta_k\|^2+4\gamma p_2\Delta\|\eta_k\|^2.
\end{aligned}\quad (27)$$

plies that the second-order dynamical local consensus over the random switching network (1) can be achieved almost surely.

Proof: The second equation in system (16) can be rewritten as follows:

$$\dot{\xi}(t) = (B_1 - A_1)\xi(t) + A_1 \int_{t-\tau}^{t} \dot{\xi}(s)ds, \tag{29}$$

where $A_1 = (W^{\mathrm{T}} \otimes I_{2n})(E[\tilde{L}] \otimes B(t))(W \otimes I_{2n})$ and $B_1 = I_{N-1} \otimes F(s_1,\ s_2,\ t)$.

Construct the Lyapunov function for system (29) as follows:

$$V(\xi(t),\ t) = \xi^{\mathrm{T}}(t)P(t)\xi(t), \tag{30}$$

where $P(t)$ also satisfies (H2). Calculating the time derivative of $V(\xi(t),\ t)$ along the trajectory of system (29) gives that

$$\begin{aligned}
&\dot{V}(\xi(t),t)\\
&= \xi^{\mathrm{T}}(t)\left[P(t)(B_1 - A_1) + (B_1 - A_1)^{\mathrm{T}}P(t) + \dot{P}(t)\right]\xi(t)\\
&+ 2\int_{t-\tau}^{t} \xi^{\mathrm{T}}(t)P(t)A_1\left[(B_1 - A_1)\xi(s) + A_1\int_{s-\tau}^{s}\dot{\xi}(\mu)d\mu\right]ds\\
&= \xi^{\mathrm{T}}(t)\left[P(t)(B_1 - A_1) + (B_1 - A_1)^{\mathrm{T}}P(t) + \dot{P}(t)\right]\xi(t)\\
&+ 2\int_{t-\tau}^{t} \xi^{\mathrm{T}}(t)P(t)A_1\left[B_1\xi(s) - A_1\xi(s-\tau)\right]ds\\
&\le \xi^{\mathrm{T}}(t)\left[P(t)(B_1 - A_1) + (B_1 - A_1)^{\mathrm{T}}P(t) + \dot{P}(t)\right]\xi(t)\\
&+ 2\left\|P(t)A_1\right\|\int_{t-\tau}^{t}\left(\|B_1\|\left|\xi^{\mathrm{T}}(t)\xi(s)\right| + \|A_1\|\left|\xi^{\mathrm{T}}(t)\xi(s-\tau)\right|\right)ds\\
&\le \xi^{\mathrm{T}}(t)\left[P(t)(B_1 - A_1) + (B_1 - A_1)^{\mathrm{T}}P(t) + \dot{P}(t)\right]\xi(t)\\
&+ \left\|P(t)A_1\right\|\int_{t-\tau}^{t}\|B_1\|\left(\xi^{\mathrm{T}}(t)\xi(t) + \xi^{\mathrm{T}}(s)\xi(s)\right)ds\\
&+ \left\|P(t)A_1\right\|\int_{t-\tau}^{t}\|A_1\|\left(\xi^{\mathrm{T}}(t)\xi(t) + \xi^{\mathrm{T}}(s-\tau)\xi(s-\tau)\right)ds.
\end{aligned} \tag{31}$$

Note $B_1 - A_1 = \bar{M}(t)$ and (H2), we can obtain the following.

$$\begin{aligned}
\dot{V}(\xi(t),t) \le &-\left(\frac{\mu}{p_2} - \frac{\tau\left\|P(t)A_1\right\|(\|B_1\| + \|A_1\|)}{p_1}\right)V(\xi(t),t)\\
&+ \frac{\tau\left\|P(t)A_1\right\|(\|B_1\| + \|A_1\|)}{p_1}\sup_{t-2\tau\le s\le t}V(\xi(s),s)\\
\le &-\left(\frac{\mu}{p_2} - \frac{\tau p_2\bar{\beta}\lambda\left(m + \bar{\beta}\lambda\right)}{p_1}\right)V(\xi(t),t)\\
&+ \frac{\tau p_2\bar{\beta}\lambda\left(m + \bar{\beta}\lambda\right)}{p_1}\sup_{t-2\tau\le s\le t}V(\xi(s),s).
\end{aligned} \tag{32}$$

Using Halanay inequality (see, e.g., [48]), we have that if $\tau < \tau^*$ there exists a unique positive real number ζ satisfies

$$\frac{\mu}{p_2} - \frac{\tau p_2\bar{\beta}\lambda\left(m + \bar{\beta}\lambda\right)}{p_1} - \zeta - \frac{\tau p_2\bar{\beta}\lambda\left(m + \bar{\beta}\lambda\right)}{p_1}\exp(2\zeta\tau) = 0, \tag{33}$$

such that $\xi(t) \le \sqrt{p_2/p_1}\|\xi(0)\|_{2\tau}\exp(-\zeta/2t)$. However, in general, the function $V(\eta(t),t)$ is not always a Lyapunov function for the stochastically switched system $\dot{\eta}(t) = M(t)\eta(t)$. If (H4) is satisfied, then there exists a $k_0 \in Z^+$ such that $t_{k_0}^* < \kappa\Delta$. For any $t \in [0, t_{k_0}^*]$, we can derive that

$$\begin{aligned}
\|\eta(t)\| \le& \|\eta(0)\| + \int_0^t \|I_{N-1}\otimes F(s_1,s_2,s)\eta(s)\\
&- (W^{\mathrm{T}}\otimes I_{2n})(\tilde{L}(\frac{s}{\Delta})\otimes B(s))(W\otimes I_{2n})\eta(s-\tau)\|ds\\
\le& \|\eta(0)\| + K\int_0^t(\|\eta(s)\| + \|\eta(s-\tau)\|)ds,
\end{aligned} \tag{34}$$

where $K = \max\{m, \bar{\beta}\lambda\}$. We therefore have

$$\|\eta(t)\| \le \|\eta(0)\| + 2K\int_0^t\left[\sup_{-\tau\le\theta\le s}\|\eta(\theta)\|\right]ds. \tag{35}$$

It follows that

$$\begin{aligned}
\sup_{-\tau\le s\le t}\|\eta(s)\| \le& \sup_{-\tau\le\theta\le 0}\|\eta(\theta)\| + \sup_{0\le s\le t}\|\eta(s)\|\\
\le& 2\sup_{-\tau\le\theta\le 0}\|\eta(\theta)\|\\
&+ 2K\int_0^t\left[\sup_{-\tau\le\theta\le s}\|\eta(\theta)\|\right]ds.
\end{aligned} \tag{36}$$

Applying the Gronwall inequality yields:

$$\sup_{-\tau\le s\le t}\|\eta(s)\| \le \left[2\sup_{-\tau\le\theta\le 0}\|\eta(\theta)\|\right]\exp(2Kt). \tag{37}$$

Thus, we can get

$$\left\|\eta(t_{k_0}^*)\right\|_\tau \le 2\|\eta(0)\|_\tau\exp\left(2Kt_{k_0}^*\right). \tag{38}$$

When $t > t_{k_0}^*$, since $V_\tau(\eta(t_{k_0}^*), t_{k_0}^*) \le p_2\left\|\eta(t_{k_0}^*)\right\|_\tau^2$ and $V_\tau(\eta(t_{k_0+1}^*), t_{k_0+1}^*) - V_\tau(\eta(t_{k_0}^*), t_{k_0}^*) \le -p_3\left|\eta(t_{k_0}^*)\right|_\tau^2 < 0$, it can be derived that

$$V_\tau(\eta(t_{k_0+1}^*), t_{k_0+1}^*) \le \left(1 - \frac{p_3}{p_2}\right)V_\tau(\eta(t_{k_0}^*), t_{k_0}^*). \tag{39}$$

Due to $0 \le V_\tau(\eta(t_{k_0+1}^*), t_{k_0+1}^*) < V_\tau(\eta(t_{k_0}^*), t_{k_0}^*)$, we have $0 \le 1 - p_3/p_2 < 1$. Repeat this computation procedure m^* (which is analytically determined in the following (41)) times, we have

$$V_\tau(\eta(t_{k_0+m^*}^*), t_{k_0+m^*}^*) \le \left(1 - \frac{p_3}{p_2}\right)^{m^*}V_\tau(\eta(t_{k_0}^*), t_{k_0}^*). \tag{40}$$

If we select

$$m^* = \left\lfloor\frac{\ln\left(\frac{p_1}{p_2}\right)}{\ln\left(1 - \frac{p_3}{p_2}\right)}\right\rfloor, \tag{41}$$

then we can get

$$\left\|\eta(t_{k_0+m^*}^*)\right\|_\tau^2 \le \left(1 - \frac{p_3}{p_2}\right)^{m^*}\frac{p_2}{p_1}\left\|\eta(t_{k_0}^*)\right\|_\tau^2, \tag{42}$$

and

$$\left(1 - \frac{p_3}{p_2}\right)^{m^*}\frac{p_2}{p_1} < 1. \tag{43}$$

Letting $\varsigma > 0$ and

$$\Delta \leq \frac{\ln\left(\frac{p_2}{p_3-p_2}\right)}{\varsigma\kappa}, \tag{44}$$

the following relationship can be established, i.e.,

$$\begin{aligned}\left(1-\frac{p_3}{p_2}\right)^{m^*} &\leq \exp(-\varsigma m^*\kappa\Delta)\\ &\leq \exp(-\varsigma(t^*_{k_0+m^*}-t^*_{k_0})).\end{aligned} \tag{45}$$

Combining (42)–(45), it can be derived that

$$\left\|\eta(t^*_{k_0+m^*})\right\|_\tau \leq \sqrt{\frac{p_2}{p_1}}\exp(-\frac{\varsigma}{2}(t^*_{k_0+m^*}-t^*_{k_0}))\left\|\eta(t^*_{k_0})\right\|_\tau. \tag{46}$$

For all $n \in Z^+$, by a similar argument procedure the same as that leading to (39), we can get

$$\|\eta(t^*_{k_0+nm^*})\|^2_\tau \leq \left(1-\frac{p_3}{p_2}\right)^{nm^*}\frac{p_2}{p_1}\|\eta(t^*_{k_0})\|^2_\tau. \tag{47}$$

By the definitions of (41) and (44), one can further has

$$\begin{aligned}\left(1-\frac{p_3}{p_2}\right)^{nm^*} &\leq \exp(-\varsigma nm^*\kappa\Delta)\\ &\leq \exp(-\varsigma(t^*_{k_0+nm^*}-t^*_{k_0}))\end{aligned} \tag{48}$$

still holds.

Therefore, $\forall n \in Z^+$, the following inequality holds.

$$\|\eta(t^*_{k_0+nm^*})\|_\tau \leq \sqrt{\frac{p_2}{p_1}}\exp(-\frac{\varsigma}{2}(t^*_{k_0+nm^*}-t^*_{k_0}))|\eta(t^*_{k_0})\|_\tau. \tag{49}$$

On the other hand, for all $t \geq t^*_{k_0}$, there exists $n_0 \in Z^+$ such that $t \in [t^*_{k_0+n_0m^*},\ t^*_{k_0+(n_0+1)m^*})$, thus we have

$$\begin{aligned}&\|\eta(t)\|\\ &\leq \left\|\eta(t^*_{k_0+n_0m^*})\right\| + \int_{t^*_{k_0+n_0m^*}}^t \|I_{N-1}\otimes F(s_1,\ s_2,\ s)\eta(s)\\ &\quad -(W^{\mathrm{T}}\otimes I_{2n})(\tilde{L}(\frac{s}{\Delta})\otimes B(s))(W\otimes I_{2n})\eta(s-\tau)\|ds\\ &\leq \left\|\eta(t^*_{k_0+n_0m^*})\right\| + K\int_{t^*_{k_0+n_0m^*}}^t (\|\eta(s)\|+\|\eta(s-\tau)\|)ds\\ &\leq \left\|\eta(t^*_{k_0+n_0m^*})\right\| + 2K\int_{t^*_{k_0+n_0m^*}}^t \|\eta(s)\|_\tau ds\\ &\leq \left\|\eta(t^*_{k_0+n_0m^*})\right\|_\tau + 2K\int_{t^*_{k_0+n_0m^*}}^t \|\eta(s)\|_\tau ds\\ &\leq \exp(2K(t-t^*_{k_0+n_0m^*}))\left\|\eta(t^*_{k_0+n_0m^*})\right\|_\tau\\ &\leq \exp(2K(t-t^*_{k_0+n_0m^*}))\\ &\quad\times\sqrt{\frac{p_2}{p_1}}\exp(-\frac{\varsigma}{2}(t^*_{k_0+n_0m^*}-t^*_{k_0}))\left\|\eta(t^*_{k_0})\right\|_\tau.\end{aligned} \tag{50}$$

By using differential theory and Gronwall inequality, we have the following. From $t - t^*_{k_0+n_0m^*}. \leq m^*\kappa\Delta$, we can get $t^*_{k_0+n_0m^*} - t^\star - t^*_{k_0} \geq t - m^*\kappa\Delta - t^*_{k_0}$. Thus, the following inequality holds.

$$\begin{aligned}\|\eta(t)\| &\leq \exp(-\frac{\varsigma}{2}(t-t^*_{k_0}))\\ &\quad\times\exp((\frac{\varsigma}{2}+2K)m^*\kappa\Delta)\sqrt{\frac{p_2}{p_1}}\left\|\eta(t^*_{k_0})\right\|_\tau.\end{aligned} \tag{51}$$

Substituting (38) into(51) yields

$$\begin{aligned}\|\eta(t)\| &\leq 2\sqrt{\frac{p_2}{p_1}}\|\eta(0)\|_\tau\\ &\quad\times\exp((\frac{\varsigma}{2}+2K)(m^*+1)\kappa\Delta)\exp(-\frac{\varsigma}{2}t).\end{aligned} \tag{52}$$

Therefore, we have that $\eta(t)$ will also decay to zero almost surely when $t \to \infty$. Therefore, the second-order dynamical local consensus over the random switching network (1) can be achieved almost surely. The proof is thus completed.

Remark 4: The classical Lyapunov approach to uniform asymptotic stability of the zero solution of dynamical system requires the existence of a positive definite, decrescent Lyapunov function whose derivative along the solutions of the system is negative definite. However, for the case of random switching topologies, the above approach may fail. On the one hand, it is difficult to construct a suitable Lyapunov function for random switching dynamic network. On the other hand, the derivative of the Lyapunov function may have positive and negative values. In this situation, we can also resort to the Lyapunov function of the time-averaged system to investigated the exponential stability (not exponentially asymptotical stability) of the corresponding random switching system. The obtained theorem additionally requires the designed Lyapunov function decrease when valuated along the solutions at an infinite subsequence of random switching moments. To reach the second-order nonlinear consensus in networks of multi-agent with directed topologies and random switching connections, a more harsh condition, i.e., at the cost of measuring position and velocity states of all agents at all times, is needed.

IV. Simulations

In this section, some numerical simulations are performed to illustrate the feasibility and effectiveness of our theoretical results presented in the previous sections. For convenience, we assume that there are totally 4 agents in the random switching directed network $G(t)$. Each agent is modeled as a second-order system with nonlinear dynamics. At time t, all agents are coupled by position and velocity states with other agents according to the topology of dynamic network $G(t)$. Suppose the agents' inner coupling matrix $B = I_3$. Through coupling, the dynamics of agent i at time t can be described by:

$$\begin{cases}\dot{x}_i(t) = v_i(t),\\ \dot{v}_i(t) = f(v_i(t)) - \alpha\sum\limits_{j=1}^4 L_{ij}(\frac{t}{\Delta})x_j(t)\\ \qquad -\beta\sum\limits_{j=1}^4 L_{ij}(\frac{t}{\Delta})v_j(t),\end{cases} \tag{53}$$

for $i = 1,2,3,4$, where $x_i = (x_{i1}, x_{i2}, x_{i3})^{\mathrm{T}}$, $v_i = (v_{i1}, v_{i2}, v_{i3})^{\mathrm{T}}$, $f(v_i) = (v_{i2}, v_{i3}, -cv_{i1} - bv_{i2} - av_{i3} + v_{i3}^2)^{\mathrm{T}}$, and a, b, c are constants. Especially, when $a = 0.44$, $b = 1.1$ and $c = 1$, the second-order isolated oscillator ($\alpha = \beta = 0$) depicts a chaotic attractor [47]. In our experiments, we use the Runge-Kutta method to solve the differential equations by letting the step size $h = 0.005$.

For the time-delay-free coupling case, it is assumed that the link weights among the agents is

$$W_e = \begin{bmatrix} 0 & 0.3301 & 0.3530 & 0.2055 \\ 0.2319 & 0 & 0.1199 & 0.2124 \\ 0.4614 & 0.2384 & 0 & 0.4771 \\ 0.1209 & 0.2344 & 0.4326 & 0 \end{bmatrix}, \quad (54)$$

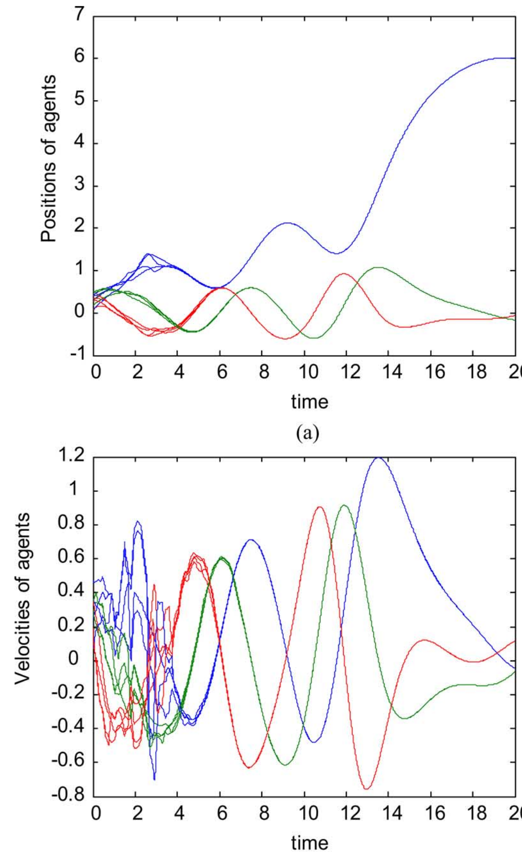
and the potential link probability among agents is

$$P = \begin{bmatrix} 0 & 0.1501 & 0.1459 & 0.0979 \\ 0.1527 & 0 & 0.3459 & 0.4887 \\ 0.4147 & 0.2204 & 0 & 0.1831 \\ 0.4853 & 0.0031 & 0.0417 & 0 \end{bmatrix}. \quad (55)$$

According to Remark 1, we choose the decomposition matrix as $W = [W_2, W_3, W_4]$, where $W_2 = [-0.0423, -0.7987, 0.36300.4780]^{\mathrm{T}}$, $W_3 = [0.7615, 0.3203, 0.1113, -0.5524]^{\mathrm{T}}$ and $W_4 = [-0.4103, 0.0971, 0.7784, -0.4651]^{\mathrm{T}}$. Obviously, $W^{\mathrm{T}}e = 0$ and $W^{\mathrm{T}}W = I_3$. By computation, we can select the parameters as follows: $\mu = 17.3058$, $m = 2.5185$, $a = 105.6677$, $\gamma = 0.9931$, $p_1 = 2$, $p_2 = 4$, $\lambda = 3.7101$, $\bar{\beta} = 1 \cdot \Delta^* = 1/2\gamma \ln((\mu - 4\gamma p_2 + a)/a) = 0.0067$. This shows that our bound is exceedingly conservative. For example, the switching rate $\Delta = 0.1$, the initial position and velocity conditions are both selected randomly over the interval $[0, 0.5]$. Fig. 1(a) and 1(b) show the time evolution process of position states and velocity states of all agents with time-delay-free couplings over the random switching directed network, respectively. It is easy to see that the second-order dynamical local consensus over the designed random switching network is achieved. Moreover, we also give the time evolution of $\mathrm{Re}(\lambda_2(L(t)))$ (the real part of the second minimum eigenvalue of $L(t)$, which is shown in Fig. 2. From Fig. 2, we can see that in some time intervals, $\mathrm{Re}(\lambda_2(L(t)))$ equals to zero, which means there exists no directed spanning trees in these topologies, i.e., there exist some isolated nodes in these topologies.

In the following, we consider a larger network with 10, 20, 30 agents, respectively. Suppose that all the link weights among the agents are 0.01 and all the potential link probabilities among agents are 0.5, and the coupling strengths $\alpha = \beta = 30$. Through simulations, we empirically find that the allowable upper bound of the switching rate will decrease as the number of the agents in the network increases when other parameters keep unchanged. Due to the limited space, we omit presenting these simulation results.

Remark 5: Before the second-order dynamical consensus is achieved, the consensus state vector $s(t)$ is unknown to us. However, we can still use one of the agents' state vectors to substitute the consensus state vector $s(t)$ to approximately estimate the time-varying parameters. In the simulations, we have used the first agent's state to replace the consensus manifold to appropriately compute the corresponding time-varying parameters.



(a)

(b)

Fig. 1. The time responses of position and velocity states of all agents with time-delay-free couplings in the random switching network. (a) Position states of all agents. (b) Velocity states of all agents.

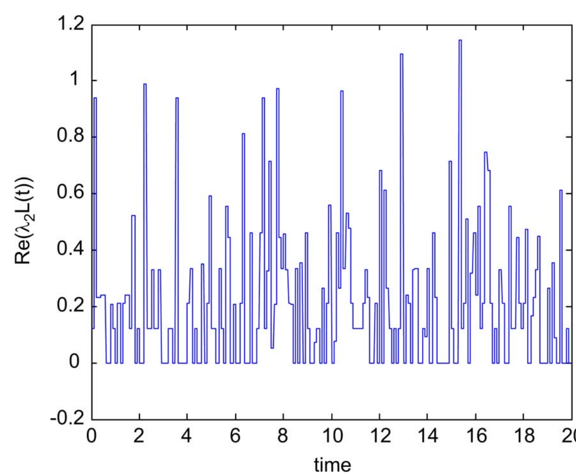


Fig. 2. The time evolution of $\mathrm{Re}(\lambda_2(L(t)))$.

For the case of time-delay coupling, the time delay is chosen as $\tau = 0.03$, other parameters are selected the same as that for the time-delay-free coupling case. Fig. 3(a) and 3(b) show the time evolution process of position states and velocity states of all agents with time-delay couplings over the random switching network. Fig. 4(a) and 4(b) display the time evolution of $V(\eta(t))$ and time evolution of logarithms with regard to $V(\eta(t))$, respectively. From Fig. 4, one can see

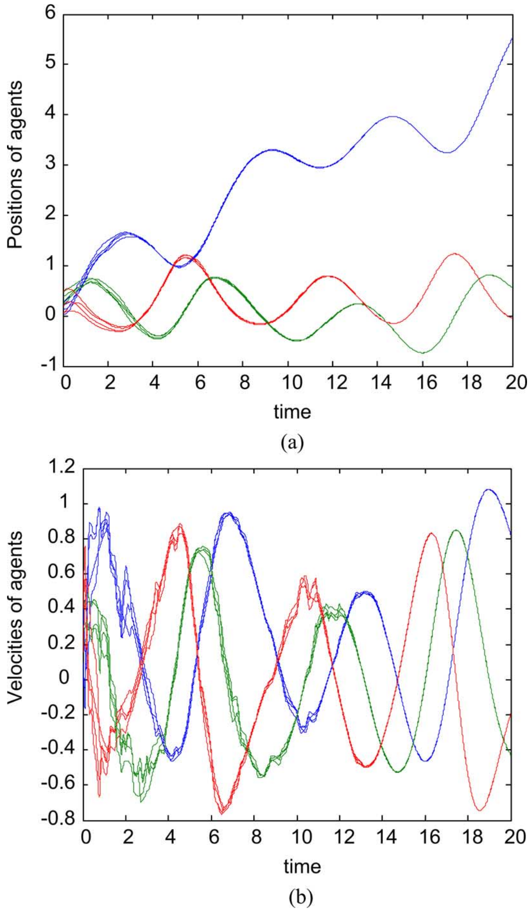


Fig. 3. The time responses of position and velocity states of all agents with time-delay couplings in the random switching network. (a) Position states of all agents. (b) Velocity states of all agents.

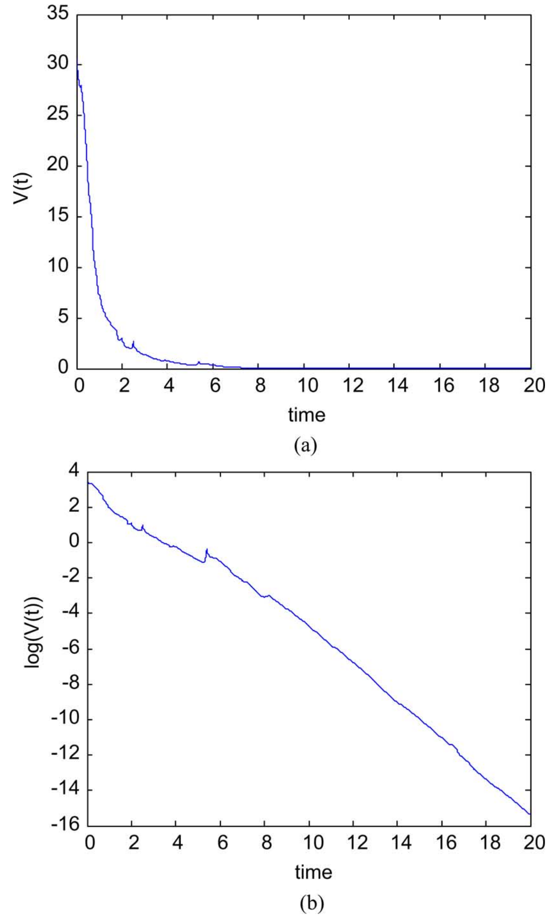


Fig. 4. (a) The time evolution of $V(\eta(t))$. (b) The time evolution of logarithms with regard to $V(\eta(t))$.

that the time derivative of $V(\eta(t))$ have positive and negative values. The reason is that the multi-agent network undergoes some topologies which do not contain directed spanning trees. However, we can observe that there exists a subsequence of the switching sequence at which the estimated values of the Lyapunov function decreases totally. In this sense, the exponential stability instead of exponentially asymptotical stability of the error dynamical system can be established.

Remark 6: The weight denotes how each agents to update the consensus algorithm. In general, we do not require W_e to have nonnegative elements. The negative weights may imply deteriorated communication channels, or natural disagreement of the child node over the information obtained from its parent node. It is noted that in our experiments, we have considered the case in which some communication links have negative weights, e.g., $W_{e13} = -0.3503$ and $W_{e43} = -0.4326$. We find that the second-order dynamical consensus over random switching network can also been achieved. Therefore, the obtained results in this paper are more practical in applications of engineering.

Remark 7: The results obtained in this paper have provided new insights about the requirements for second-order dynamical consensus when the network topology is random switching. The results also show that even if the network is not always connected instantaneously, sufficient information is propagated through the network to allow almost sure consensus as long as the network which corresponds to the fixed time-averaged topology has a directed spanning tree, and that the switching rate is sufficiently fast. We note that the results obtained in this paper are only limited to the second-order local dynamical consensus.

Remark 8: From the above simulation, we found that the allowable time delay and switching rate may be significantly bigger than the theoretically estimated values. It seems that our estimations of τ^* and Δ^* are somewhat conservative. Since we hope that the resulting system is exponentially stable with the convergence rate ς satisfying (45), within this framework the estimations can hardly be improved. The limitations on time delay and the switching rate are a commonly unsolved problem which deserves further investigations.

V. Conclusions

In this paper, the problem of second-order dynamical consensus over the random switching directed networks has been studied in detail. Note that our theoretical results are only limited to the local consensus, despite to the fact that some numerical results seem to indicate that the stronger property of global consensus might in fact be exhibited by the case-study system. The orthogonal decomposition method is used to simplify the theoretical analysis. We pose the second-order nonlinear consensus problem in a stochastic framework where the communication among the agents is modeled as weighted directed random switching graph. The theoretical results show that the local consensus can be achieved almost surely if the time-averaged communication network supports the consensus and the time delay and switching rate are less than two upper bounds respectively, which are estimated analytically. The obtained results are quite powerful, and can be further used to solve various switching cases for complex dynamical networks. In this framework of random switching networks, the following issues deserve careful studies: i) consensus of agents with different nonlinear dynamics; ii) consensus of agents with time-varying delay couplings; iii) cluster consensus; iv) consensus with the communications constraints, such as packet losses, channel noises, limited width, *ect*. These problems will be discussed in future papers.

ACKNOWLEDGMENT

The authors are very grateful to the associate editor and anonymous reviewers for their constructive comments based on which the presentation of the paper has been greatly improved.

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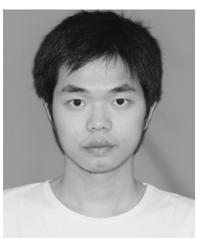

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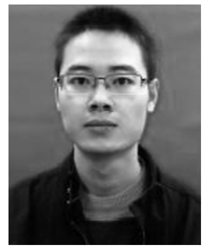

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